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CLASSICAL ABELIAN THEORIES OF MONOPOLES AND DYONS

BALWANT SINGH RAJPUT* AND SANDEEP KUMAR SINGH**

**I-11 Gamma-2, Greater Noida (UP) (India), Gmail: bsrajp@gmail.com*

***J-101, Gamma-2, Greater Noida (UP) (India), Gmail:Sandeep_kumar_rajput@hotmail.com*

Constructing the Lagrangian density for spin-1 generalized charge (bosonic dyon) in abelian theory, the generalized of motion has been derived and the gauge invariant and rotationally symmetric orbital angular momentum of a dyon moving in the field of other dyon has been constructed and it has been shown that each dyon carries a residual angular momentum which leads to chirality quantization condition.

INTRODUCTION

The question of existence of monopoles anddyons has gathered enormous potential importance in connection with the issue of quark confinement] ,magnetic condensation of vacuum,] as possible explanation of CP violation, their role in catalyzing proton decay and in the structure of black holes and also in the unification of gravitation with generalized electromagnetic fields. In the present paper the gauge invariant and consistent classical theories of Abelian as well as non-Abelian monoples and dyons have been revisited in view of possible symmetry of Maxwell's field equations and it has been shown that the mere existence of magnetic charge implies the quantization of electric charge, emphasizing that there are no theoretical reasons to exclude the existence of monopoles and dyons. Constructing the Lagrangian density for spin-1 generalized charge (bosonic dyon) in abelian theory, the generalized equation of motion has been derived and the gauge invariant and rotationally symmetric orbital angular momentum of a dyon moving in the field of other dyon has been constructed and it has been shown that each dyon carries a residual angular momentum which leads to chirality quantization condition. For each pair of dyons, this residual angular momentum has been shown to generate a one-dimensional representation of the pair of four-momenta associated with these particles to lead to chirality dependent multiplicity in the eigen values of angular momentum of an Abelian dyon.

CLASSICAL ABELIAN THEORY OF MONOPOLES

It is quite surprising, rather disturbing, that the symmetry between electric and magnetic fields which led Maxwell to unify these fields into electromagnetism, does not seem to be realized in Maxwell's field equations given as follows in terms of electric charge density j_0 and electric current density $\vec{j}$

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$$\text{div } \vec{E} = j_0,$$
$$\text{div } \vec{H} = 0,$$
$$\text{curl } \vec{E} = -\frac{\partial \vec{H}}{\partial t} \quad \text{... (2.1)}$$
$$\text{curl } \vec{H} = \vec{j} + \frac{\partial \vec{E}}{\partial t}$$

in rational Gaussian unit system ($\hbar$ = c = 1) which will be used throughout this work. These equations are not symmetrical between electricity and magnetism and also not invariant under duality transformations.

$$\vec{E} \to \vec{H};$$
$$\vec{H} \to -\vec{E} \quad \text{... (2.2)}$$

Moreover, these equations show that magnetic field is always transverse to direction of propagation while electric field has longitudinal component also, which is proportional to electric charge source density.

The lack of symmetry in equations (2.1) led Dirac[1,2] to put forward the idea of magnetic monopole as the magnetic analogue of electric charge. This remarkable suggestion of Dirac revealed the explanation of quantization of electric charge and led to natural generalization of electrodynamics in terms of following equations

$$\text{div } \vec{E} = j_0,$$
$$\text{div } \vec{H} = k_0,$$
$$\text{curl } \vec{E} = -\frac{\partial \vec{H}}{\partial t} - \vec{k} \quad \text{... (2.3)}$$
$$curl\, \vec{H} = \frac{\partial \vec{E}}{\partial t} + \vec{j}$$

where $\vec{j}$, j_0 and $\vec{k}, k_0$ are spatial and temporal parts of electric and magnetic four-current densities. j_μ and k_μ respectively. These equations may also be written as

$$F_{\mu\nu,\nu} = j_\mu, \quad \text{... (2.4)}$$
$$F^*_{\mu\nu,\nu} = k_\mu \quad \text{... (2.5)}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor and $F^*_{\mu\nu}$ is its dual tensor defined as

$$F^*_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}F^{\lambda\rho} \quad \text{... (2.6)}$$

with $\varepsilon_{\mu\nu\lambda\rho}$ as completely antisymmetric Ricci tensor. Equations (2.2) and (2.3) are symmetrical under the duality transformations.

$$F_{\mu\nu} \to F^*_{\mu\nu}, \; F^*_{\mu\nu} \to -F_{\mu\nu} \quad \text{... (2.7)}$$

along with

$$j_\mu \to k_\mu, \; k_\mu \to -j_\mu \quad \text{... (2.8)}$$

In Dirac's theory such a symmetric formalism was obtained quantum mechanically, where wave function had a non-integral (or path independent) phase factor. His work led to the profound theoretical consequences of the existence of magnetic monopole at quantum level. He demonstrated that a dually symmetric electromagnetic theory could be quantized provided the following condition is satisfied for the electric charge *e* and magnetic charge *g* in the theory;

$$eg = \frac{1}{2} n \qquad \text{... (2.9)}$$

where *n* is an integer. It is celebrated Dirac quantization condition derived by assuming that a particle has either an electric or a magnetic charge. The similar result was obtained by Saha[3,4] independently from the very simple consideration of classical electrodynamics by taking charge and monopole at two different points and computing the angular momentum of the system about the line joining two points. This derivation is given here in very simple manner by considering a particle of mass *m* and electric charge *e*, moving in the field

$$\vec{H} = g(\vec{r} / r^3) \qquad \text{… (2.10)}$$

of monopole of strength *g* fixed at the origin. The equation of motion of the particle carrying electric charge *e* is given by

$$m\ddot{\vec{r}} = e\ (\dot{\vec{r}} \times \vec{H}) \qquad \text{... (2.11)}$$

where magnetic field $\vec{H}$ is spherically symmetric and therefore one can expect some thing like conservation of angular momentum. However, it is not the orbital angular momentum which is a conserved quantity here. Rather, the rate of change of orbital angular momentum is given by

$$\frac{d}{dt}(\vec{r} \times m\dot{\vec{r}}) = \vec{r} \times m\ddot{\vec{r}} = \frac{eg}{r^3}\vec{r} \times (\dot{\vec{r}} \times \vec{r}) = \frac{d}{dt}(eg\hat{r}) \qquad \text{... (2.12)}$$

where $\hat{r}$ is unit vector along $\vec{r}$

This result suggests that the conserved total angular momentum should be defined as

$$\vec{L} = \vec{r} \times m\dot{\vec{r}} - eg\hat{r} \qquad \text{… (2.13)}$$

where the second term can be interpreted as the contribution of electromagnetic field and it may be obtained by integrating the moment of Poynting vector $(\vec{E} \times \vec{H})$ over whole space;

$$\vec{L}_{em} = \int d^3 x[\vec{x} \times (\vec{E} \times \vec{H})] \qquad \text{... (2.14)}$$

where $\vec{H}$ is the radial field given by equation (2.8) and $\vec{E}$ is the field due to electric charge at $\vec{r}$. Thus

$$L_{em}^{(i)} = \int d^3 x E^j (\delta_{ij} - \hat{x}^i \hat{x}_j) \frac{g}{x}$$

$$= \int d^3 x E^j \frac{\partial}{\partial x_j} (g x^i),$$

$$= -\int d^3 x (\nabla.\vec{E}) g \hat{x}^i, \quad \text{... (2.15)}$$

integrating by parts.

It gives

$$\vec{L}_{em} = -\, eg\ \hat{r}, \quad \text{... (2.16)}$$

$$\nabla.\vec{E} \;\; = e\delta(\vec{x} - \vec{r})$$

Thus the total conserved angular momentum of the system is the sum of orbital angular momentum of the particle and the angular momentum caused by the electromagnetic field. Its conservation means that the momentum passes back and forth between the particle and the field in the presence of electric and magnetic charges. The radial component of total angular momentum given by equation (2.13) is

$$\hat{r}.\hat{L} = -eg$$

On quantization, we expect the components of $\vec{L}$ to have half integral eigen values (in the units of $\hbar$) and hence we get

$$eg = ½\, n \quad \text{... (2.17)}$$

which is Dirac quantization condition (2.9). It leads to the following inferences.

(i) Mere existence of isolated magnetic charge implies the quantization of electric charge.

(ii) There are no theoretical reasons which exclude the existence of magnetic charge.

Dirac quantization condition gave rise to lot of literature on monopoles but there have been certain difficulties, which are encountered in Dirac's theory. For instance, let us consider the magnetic field produced by a magnetic monopole of charge g located at origin and describe it by vector potential $\vec{A}$. Then we have

$$\vec{H} \neq \text{curl}\ \vec{A} \quad \text{… (2.18)}$$

along the line going from monopole to infinity. Such a line, may be curved or planer, is referred as Dirac string in literature. For the straight string $S^{(n)}$. we may write

$$\vec{A}^n(r) = \frac{g}{r} \frac{\vec{r} \times \hat{n}}{(r - \vec{r}.\hat{n})} \quad \text{… (2.19)}$$

where $\hat{n}$ is the direction vector of string $S^{(n).}$ On the string we get

$$[A^{(n)}] = \infty \quad \text{… (2.20)}$$

and hence in Dirac theory a string of arbitrary shape ends at each monopole, the potential $\vec{A}$ is singular along the string and a charged particle can never pass through such strings. These conditions are referred as Dirac's veto which has been the basic cause of troubles in the development of a consistent theory of magnetic monopoles. Singular electromagnetic potential is not present in the usual field theory and this veto gave rise to difficulties in scattering of electrons and monopoles. It has been shown[5,6] that the Hamiltonian for an electron in the field of monopole is not Hermitian because of Dirac veto. Although an attempt was made[7,8] to explain the occurrence of string in Dirac's theory by demonstrating that non-Abelian vortices must contain a single unit of quantized flux absorbed by Dirac monopole at each end but the string remained the topic of criticism and controversies since the original work of Dirac.

The first attempt to construct a theory of monopoles, free from Dirac's veto, was made by Mandelstam[9] and Cabbibo and Ferrari[10] by developing gauge independent method using two vector potentials. Wu and Yang[11,12] introduced fiber bundle method into gauge theories and reformulated Dirac's theory to avoid any singularity in $\vec{A}$ by dividing the space surrounding the monopoles into few regions whose vector potentials are connected through gauge transformations. In this approach singular potential of Dirac's theory is replaced by two potentials. This approach is not useful when we have to deal with quantum field theory. Another approach of Keon[13] based on idea of Cabbibo and Ferrari, free from Dirac's veto, lacks in action principle in the presence of particles carrying both electric and magnetic charges (dyons).

CLASSICAL ABELIAN THEORY OF DYONS

The theory of pure monopoles suffers from many paradoxes like Dirac's veto and wrong connection between spin and statistics[14]. These problems could be solved by considering electric and magnetic charges on the same particle (dyon)[15]. Schiwinger[16-18] formulated a relativistic covariant quantum field theory of point-like particles with either electric or magnetic charge and sharpened[19] Dirac quantization condition as

$$e_g = n \qquad \text{... (3.1)}$$

Expanding this quantum field theory of spin – ½ magnetic charges to the field associated with the particles carrying electric and magnetic charges simultaneously, Zwanziger[20] slowed that if two particles have magnetic charges g_1 and g_2 in addition to their electric charges e_1 and e_2 with electric and magnetic coupling parameters

$$\alpha_{12} = e_1 e_2 + g_1 g_2$$

and
$$\mu_{12} = e_1 g_2 - g_1 e_2 \qquad \text{... (3.2)}$$

respectively, then Hamiltonian possesses the same higher symmetry as that of pure Coulmbian problem. He derived the following chirality quantization condition

$$\mu_{12} = e_1 g_2 - g_1 e_2 = \tfrac{1}{2} n \qquad \text{... (3.3)}$$

where n is an integer. But this theory does not show a rotational invariance when gauge is massive. Later formulations proposed by Blagojevic *et al*[21] as an improvement of Schwinger's and Zwanzigar's, theories, lack in Lorentz invariance while in another attempt of Brandt *et al*[22] to maintain it, the use of controversial string variables has been made.

A gauge invariant and Lorentz covariant quantum field theory of fields associated with dyons has been developed[23-25] in purely group theoretical manner by using two four-potentials and assuming the generalized charge, generalized current and generalized four-potential as complex quantities with their real and imaginary parts as electric and magnetic constituents i.e.

generalized charge $q = e - i_g$, ... (3.4)

generalized four-current $J_\mu = j_\mu - ik_\mu$ ... (3.5)

and generalized four-potential $V_\mu = A_\mu - iB_\mu$ ... (3.6)

where e and g are electric and magnetic charges on dyon; j_μ and k_μ are electric and magnetic four-current densities and $\{A_\mu\}$ and $\{B_\mu\}$ are the electric and magnetic four-potentials associated with dyons. Taking the wave function associated with generalized fields as

$$\vec{\psi} = \vec{E} - i\vec{H}$$

The generalized field equations of these fields may be written as

$$\nabla.\vec{\psi} = J_0 \quad \text{... (3.7)}$$

and $$\nabla \times \vec{\psi} = -i\vec{j} - i\frac{\partial\vec{\psi}}{\partial t} \quad \text{... (3.8)}$$

where J_0 and $\vec{J}$ are the temporal and spatial parts of J_μ defined by equation (3.5). In the compact form these equations may be written as

$$G_{\mu\nu,\nu} = J_\mu$$

and $$G^d_{\mu\nu,\nu} = 0 \quad \text{... (3.9)}$$

where $G_{\mu\nu}$, the generalized field tensor, is given as

$$G_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \quad \text{... (3.10)}$$

and $G^d_{\mu\nu}$ is its dual given as

$$G^d_{\mu\nu} = \frac{1}{2}\in_{\mu\nu\alpha\beta} G^{\alpha\beta} \quad \text{... (3.11)}$$

Equation (3.10) may also be written as

$$G_{\mu\nu} = F_{\mu\nu} - iH_{\mu\nu} \quad \text{... (3.12)}$$

where $$F_{\mu n} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{... (3.13)}$$

and $$H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \quad \text{... (3.14)}$$

Then equations (3.9) reduce to the following form

$$F_{\mu\nu,\nu} = j_\mu \qquad \text{... (3.15)}$$

and $$H_{\mu\nu,\nu} = k_\mu \qquad \text{... (3.16)}$$

These equations are symmetrical under the duality transformations

$$F_{\mu\nu} \to H_{\mu\nu}; \quad H_{\mu\nu} \to -F_{\mu\nu};$$
$$j_\mu \to k_\mu; \quad k_\mu \to -j_\mu \qquad \text{... (3.17)}$$

In terms of generalized four-potential given by equation (3.6), the field equations (3.9) may be written in the following standard form by using relation (3.10);

$$V_\mu = J_\mu \qquad \text{... (3.18)}$$

The Lagrangian density for spin-1 generalized charge (*i.e.* bosonic dyon) of rest mass m_o may be written as follows in the Abelian theory;

$$L = m_0 - \frac{1}{4}[\alpha\{(A_{\nu,\mu} - A_{\mu,\nu})^2 - (B_{\nu,\mu} - B_{\mu,\nu})^2\} - 2\beta\{(A_{\nu,\mu} - A_{\mu,\nu})(B_{\nu,\mu} - B_{\mu,\nu})\}$$
$$+ \{(\alpha A_\mu - \beta B_\mu) j_\mu - (\alpha B_\mu + \beta A_\mu) k_\mu\}]$$
$$= L_p + L_f + L_I \qquad \text{... (3.19)}$$

where α and β are real positive unimodular parameters *i.e.*,

$$|\alpha|^2 + |\beta|^2 = 1 \qquad \text{... (3.20)}$$

L_P, L_f and L_I are free particle, field and interaction Lagrangians respectively. The action integral may be written as

$$S = \int_{t_1}^{t_2} L dt = S_p + S_f + S_I \qquad \text{... (3.21)}$$

Varying the trajectory of particle without changing the field, we get the following equation of motion

$$m\ddot{x}_\mu = \text{Re}\,(q^* G_{\mu\nu})\, u^\nu \qquad \text{... (3.22)}$$

where Re denotes the real part and u^ν is the ν^{th} component of four-velocity of dyon. It may also be written as

$$m\ddot{x}_\mu = (eF_{\mu\nu} + gH_{\mu\nu})\, u^\nu \qquad \text{... (3.22a)}$$

and $$\frac{H_{\mu\nu}}{F_{\mu\nu}} = \frac{g}{e} \qquad \text{.... (3.22b)}$$

An Abelian dyon, moving in the generalized field of another dyon, carries a residual angular momentum[26] (field contribution) besides its orbital and spin angular moment. If we consider i^{th} Abelian dyon moving in the field of j^{th} dyon (assumed as rest), the gauge invariant rotationally symmetric orbital angular momentum vector may be written as[26].

$$\vec{J} = \vec{r} \times (\vec{p} - \mu_{ij} \vec{V}^T) + \mu_{ij} \frac{\vec{r}}{r} \quad ... (3.23)$$

where $\vec{r}$ is the position vector, $\vec{p}$ is the linear momentum of i^{th} dyon, $\vec{V}^T$ is the transverse generalized vector potential of the field associated with j^{th} dyon and μ_{ij} is the magnetic coupling parameter defined as

$$\mu_{ij} = e_i g_j - e_j g_i \quad ... (3.24)$$

The last term in equation (3.23) is the residual angular momentum carried by i^{th} dyon besides its usual orbital angular momentum and spin-angular momentum.

$$\vec{J}_{res} = \mu_{ij} \frac{\vec{r}}{r} \quad … (3.25)$$

For each pair of dyons, this residual angular momentum generates a one dimensional representation of the pair of four-momenta associated with these particles. This is the subgroup of the Lorentz group which leaves both four-momenta invariant. This residual angular momentum leads to chirality dependent multiplicity in the eigen values of angular momentum of an Abelian dyon[27].

DISCUSSION

Dirac's quantization condition, given by eqn. (2.15), shows that the mere existence of the magnetic charge implies the quantization of electric charge and emphasizes that there cannot be any theoretical reasons to exclude the existence of monopoles. Equations (3.2) give the electric and magnetic coupling parameters for the system of two dyons. Generalized field equations (3.9) have been constructed for the generalized fields associated with Abelian dyons with the generalized charges defined by equation (3.4) as complex quantity with its real and imaginary parts as electric and magnetic charges respectively. The generalized field tensor for these fields have been constructed in equation (3.10) in terms of derivatives of generalized potential defined by equation (3.6) as a complex quantity with its real and imaginary parts as electric and magnetic constituents. The generalized field tensor has also been constructed as a complex quantity in equation (3.12) with its real and imaginary parts as electric and magnetic constituents satisfying field equations (2.15) and (2.16) in terms of electric and magnetic four-current densities j_μ and k_μ which constitute the generalized four-current density in equation (3.5). These field equations are symmetrical under the duality transformations given by equation (3.17). Lagrangian field density, given by equations (3.19), reproduces the field equations (3.15) and (3.16) with the variations with respect to electric and magnetic four-potentials A_μ and B_μ respectively. This Lagrangian also leads to equation of motion (3.22). Gauge invariant and rotationally symmetric orbital angular momentum of an Abelian dyon moving in the field of another Abelian dyon has been constructed in equation (3.23) and it has been shown that each dyon carries a residual angular momentum given by equation (3.25), which leads to chirality quantization condition $\mu_{ij} = \frac{1}{2} n$, where *n* is an integer and μ_{ij} is the magnetic coupling parameter defined by equation (3.24). For each pair of dyons, this residual angular momentum generates a one-dimensional representation of the pair of four-momenta associated with these particles. This residual angular momentum also leads to chirality dependent multiplicity in the eigen values of angular momentum of an Abelian dyon. Such a residual angular momentum has been used[28,29] to find the large number of monopole operators with equal scaling dimensions and a wide range of spins and flavor symmetry irreducible representations. The electric-magnetic symmetry, established in second and third sections of this paper has been shown[30-32], as most fundamental which extends to full quantum behavior leading monopole to form a quantum Bose condensate dual to charge cooper pair condensate in superconductors.

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