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FIXED POINT THEOREM IN *b*-METRIC SPACE WITH DIFFERENT CONTRACTIVE MAPPING

C. SREEDHAR

NBKR Institute of Science and Technology, Vidya Nagar-524413
Andhra Pradesh State, India

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In this paper we present the completeness and uniqueness of fixed point on b-metric space which extend the known results of fixed point theorems.

INDEX KEY : Common field point, b-metric space AMS subject classification : 47H10, 54H2S.

INTRODUCTION

Backhtin introduced the concept of *b*-metric space in 1989, results on *b*-metric space were extended by Czerwik [1, 5], Mehmet kir [6], Boriceanu [4], Bota [3] and Pacurer [7].

SOME PRELIMINARY RESULT

Definition 2.1. Let X be a non-empty set and $s \geq 1$ be the given real number. A mapping $d : X \star X \to \Re^+$ is called a *b*-metric iff

2.1.1 $d(x, y) = 0$ if and only if $x = y$

2.1.2 $d(x,y) = d(y, x)$

2.1.3 $d(x, z) \leq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$ and $s \geq 1$ where s is a real number.

The pair (X, d) is called a *b*-metric space.

Definition 2.2. A sequence $\{x_n\}$ is called a Cauchy sequence in *b*-metric space (X, d) if all $\varepsilon > 0$ $d(x_n, x_m) < \varepsilon$ for each $m, n \geq n(\varepsilon) \in N$.

Definition 2.3. A sequence $\{x_n\}$ in *b*-metric space (X, d) is called a convergent sequence if $d(x_n, x) < \varepsilon$ for all $\varepsilon > 0$ and $n \geq n(\varepsilon)$ where $n(\varepsilon) \in N$ (Set of natural numbers).

Definition 2.4. *b*-metric space is said to be complete *b*-metric space if every Cauchy sequence is convergent.

MAIN RESULT

Theorem 2.1. Let (X, d) be a complete *b*-metric space. Let T be a mapping $T : X \to X$ such that

$$d(Tx, Ty) \leq a(d(x, y) + d(x, Ty) + b \max\{d(x, Tx), d(x, Ty), dy(x, Tx)\} \quad \text{... (1)}$$

where $a, b > 0$ such that $a(1 + 2s) + b < 1$ for every $x, y \in X$ and $s \geq 1$. Then T has a unique fixed point.

Proof : Let $x_0 \in X$ and x_n be a sequence in X defined by recursion

$$x_n = Tx_{n-1} = T^n x_0 \qquad n = 1, 2, 3, 4 \qquad \text{... (2)}$$

By (1) and (2) we obtain that

$d(x_n, x_{n+1}) \leq a\,(d\,(x_{n-1}, x_n) + sa\,(d\,(x_{n-1}, x_n) + d\,(x_n\, x_{n+1})) + b \max\{d\,(x_{n-1}, x_n), d\,(x_n, x_{n+1})\}$

$d\,(x_n, x_{n+1}) \leq a\,(d\,(x_{n-1}, x_n) + sa\,\{d\,(x_{n-1}, x_n) + d\,(x_n, x_{n+1})\} + bM_1$

where $M_1 = \max d\,(x_{n-1}, x_n), d\,(x_n, x_{n+1})$

Case 1. If suppose that $M_1 = d\,(x_{n+1}, x_n)$ then we have $M_1 = d\,(x_{n+1}, x_n)$ then we have

$d\,(x_n, x_{n+1}) \leq ad\,(x_{n-1}, x_n) + as\,d\,(x_{n-1}, x_n) + asd\,(x_n, x_{n+1}) + b\,d\,(x_{n+1}, x_n)$

$d(x_n, x_{n+1})\,(1 - as - b) \leq (a + as)\,d\,(x_{n-1}, x_n)$

$$d(x_n, x_{n+1}) \leq \frac{(a+as)}{(1-as-b)} d\,(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}) \leq kd\,(x_{n-1}, x_n) \text{ where } k = \frac{a+as}{1-as-b} < 1$$

$$d(x_n, x_{n+1}) \leq k^2 d(x_{n-2}, T_{n-1})$$

Continuing this process we get $d(x_n, x_{n+1}) \leq k^n d\,(x_0, x_1)$

Case 2. If suppose $M_1 = d\,(x_{n-1}, x_n)$

$$d(x_n, x_{n+1}) \leq a\,d\,(x_{n-1}, x_n) + as\,[d\,(x_{n-1}, x_n) + d(x_n, x_{n+1})] + b\,d(x_{n-1}, x_n)$$

$$(1-as)\,d(x_n, x_{n+1}) \leq (a+as+b)\,d\,(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}) \leq kd\,(x_{n-1}, x_n) \text{ where } k = \frac{(a+ab)}{(1-as)} < 1$$

$$d(x_n, x_{n+1}) \leq k^2 d\,(x_{n-1}, x_n)$$

Continuing this process we get $d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$

Thus T is a contractive mapping. Now we show that $\{x_n\}$ is a Cauchy sequence in X.

Let $m, n \in \aleph$, $(m > n)$

$$d\,(x_n, x_m) \leq s[d\,(x_n, x_{n+1}) + d\,((x_{n+1}, x_m\,|)]$$

$$d\,(x_n, x_m) \leq s\,d\,(x_n, x_{n+1}) + s\,d\,(x_{n+1}, x_m)$$

$$d\,(x_n, x_m) \leq s[d\,x_n, x_{n+1}) + s^2 d\,((x_{n+1}, x_{n+2}) + s^2 d\,((x_{n+2}, x_m)]$$

$$d\,(x_n, x_m) \leq sk^n[d\,(x_0, x_1)] + s^2 k^{n+1} d\,((x_0, x_1) + s^3 k^{n+2} d\,((x_0, x_1)] + ...$$

$$d\,(x_n, x_m) \leq sk^n[d\,(x_0, x_1)[1 + sk + s^2k^2 + ...]$$

Then $\lim d(x_n, x_m) = 0$ as $m, n \to \infty$, since $k < 1$

$$\lim_{x\to\infty} \frac{sk^n}{1-sk} d\,(x_0, x_1) = 0$$

Hence $\{x_n\}$ is a Cauchy sequence in X. $\{x_n\}$ converges to $x \in X$.

Now we show that $x\star$ is a fixed point of T.

$d\,(x\star, Tx\star) \leq s\,[d\,(x\star, x_{n+1}) + d\,(x_{n+1}, Tx\star)]$

$d\,(x\star, Tx\star) \leq s\,[d\,(x\star, x_{n+1}) + d\,(Tx_n, Tx\star)]$

$d\,(x\star, Tx\star) \leq s\,[d\,(x\star, x_{n+1}) + d\,(Tx_n, Tx\star)]$

$\leq s[d\,(x\star, x_{n+1})] + as\,d\,(x_n, x\star) + asd\,(x_n, Tx\star) + bs \max\{d\,(x_n, x_{n+1}),$

$d\,(x_n, Tx\star), d\,(x\star, x_{n+1})\}$

$d(x\star, Tx\star) \le s\,[d(x\star, x_{n+1})] + as\, d(x_n, x\star) + asd(x_n, Tx\star)$
$+ bs \max\{d(x_n, x_{n+1}), d(x_n, Tx\star), d(x\star, x_{n+1})\}$

$d(x\star, Tx\star) \le s\,[d(x\star, x_{n+1})] + as\, d(x_n, Tx\star) + bsM_2$

where $M_2 = \{d(x_n, x_{n+1}), d(x_n, Tx\star), d(x\star, x_{n+1})\}$

Case 1 : If $M_2 = d(x_n, x_{n+1})$

$d(x\star, Tx\star) \le s\,[d(x\star, x_{n+1})] + asd(x_n, x\star) + asd(x_n, Tx\star) + bsd(x_n, x_{n+1})$

$d(x\star, Tx\star) \le s\,[d(x\star, x_{n+1})] + as\,[sd(x_n, Tx\star)] + as\,[sd(x\star, Tx\star)] + asd(x_n, Tx\star)$
$+ bsd(x_n, x_{n+1})$

$d(x\star, Tx\star) \le s\,[d(x\star, x_{n+1})] + as^2 d(x_n, Tx\star) + as^2\, d(x\star, Tx\star) + asd(x_n, Tx\star)$
$+ bs^2 d(x_n, x\star) + bs^2 d(x\star, x_{n+1})$

$(1 - s^2 a)\, d(x\star, Tx\star) \le (s + s^2 b)\, d(x\star, x_{n+1}) + (a^2 s + as)\, s\,[d(x_n, x\star)$
$+ d(x\star, Tx\star)] + bs^2 d(x_n, x\star)$

$(1 - s^2 a - s^2 a^2 - as^2)\, d(x\star, Tx\star) \le (s + s^2 b)\, d(x\star, x_{n+1}) + (a^2 s^2 + as^2 + bs^2)\, d(x_n, x\star)$

$$d(x^*, Tx^*) \le \frac{s + s^2 b}{1 - 2as^2 - a^2 s^2}\, d(x^*, x_{n+1}) + \frac{a^2 s^2 + as^2 + bs^2}{1 - 2as^2 - a^2 s^2}\, d(x_n, x^*)$$

Taking limit $n \to \infty$ we get $d(x\star, Tx\star) = 0$

$$x\star = Tx\star$$

Therefore $x\star$ is a fixed point of T.

Case 2 : Suppose if $M_2 = d(x\star, Tx\star)$

$d(x\star, Tx\star) \le sd(x\star, x_{n+1}) + sd(x_{n+1}, Tx\star) = sd(x\star, x_{n+1}) + sd(Tx_n, Tx\star)$
$\le sd(x\star, x_{n+1}) + s\,[a\,(d(x_n, x\star) + d(x_n, Tx\star)] + sb\, d(x\star, Tx\star)$

$d(x\star, Tx\star) \le sd(x\star, x_{n+1}) + as\, d(x_n, x\star) + as^2\,(d(x_n, x\star) + d(x\star, Tx\star)) + sbd(x\star, Tx\star)$

$(1 - as^2 - sb)\, d(x\star, Tx\star) \le sd(x\star, x_{n+1}) + (as + as^2)\, d(x_n, x\star)$

$$d(x^*, Tx^*) \le \frac{s}{1 - 2as^2 - a^2 s^2}\, d(x^*, x_{n+1}) + \frac{as + as^2}{1 - 2as^2 - sb}\, d(x_n, x^*)$$

Taking limit as $n \to \infty$ we get $d(x\star, Tx\star) = 0$

Therefore $x\star = Tx\star$

$x\star$ is a fixed point of T.

Case 3 : Suppose if $M_2 = d(x\star, x_{n+1})$

$d(x\star, Tx\star) \le sd(x\star, x_{n+1}) + sd(x_{n+1}, Tx\star) = sd(x\star, x_{n+1}) + sd(Tx_n, Tx\star)$
$\le sd(x\star, x_{n+1}) + s\,[a\,(d(x_n, x\star) + d(x_n, Tx\star)] + sbd(x\star, x_{n+1})$

$d(x\star, Tx\star) \le (s + sb)\, d(x\star, x_{n+1}) + asd(x_n, x\star) + as^2\,[d(x_n, x\star) + d(x\star, Tx\star)]$

$(1 - s^2 a)\, d(x\star, Tx\star) \le (s + sb)\, d(x\star, x_{n+1}) + (as + as^2)\, d(x_n, x\star)$

$$d(x^*, Tx^*) \le \frac{s + sb}{1 - as^2}\, d(x^*, x_{n+1}) + \frac{as + as^2}{1 - as^2}\, d(x_n, x^*)$$

Taking limit as $n \to \infty$ we get $d(x\star, Tx\star) = 0$

Therefore $x\star = Tx\star$

$x\star$ is a fixed point of T.

UNIQUENESS OF FIXED POINT

We have to show that $x\star$ is a unique fixed point of T. Assume that x' is another fixed point of T. Then we have $Tx' = x'$

and $d(x\star, x') = d(Tx\star, Tx') \leq a(d(x\star, x') + d(x\star, Tx') + b \max\{d(x\star, Tx\star), d(x', Tx'), d(x', Tx\star)\}$

$$d(x\star, x') \leq a(d(x\star, x') + d(x\star, x') + b \max\{d(x\star, x\star), d(x', x'), d(x', x\star)\}$$

$$d(x\star, x') \leq (2a + b)\, d(x\star, x')\}.$$

This is a contradiction. Therefore $x\star = x'$.

This completes the proof. Hence $x\star$ is the unique fixed point of T.

Example 2.1 : Let $X = [0, 1]$ and $d : X \star X \to [0, \infty)$ is defined by $d(x, y) = = \frac{|x-y|}{2}$

and the mapping $T : X \to X$ defined by $Tx = \begin{cases} \frac{x}{2} \text{ if } & x \neq 0 \\ 0 \text{ if } & x = 0 \end{cases}$

It satisfies the contraction condition

$$d(Tx, Ty) \leq a(d(x, y) + d(x, Ty) + b \max d(x, Tx), d(x, Ty), d(y, Tx)$$

for $a = 2$, $b = 3$ and $s = \frac{4}{3}$. Then 0 is the fixed point of the mapping T.

REFERENCES

1. Czerwik, S., Contraction mappings in *b*-metric spaces, *Acta Mathematica et Informatica Universities Ostraviensis*, **1**, 5-11 (1993).
2. Banach, S., Surles operations dans les ensembles abstract et leur application aux equation integrals, *Fund. Math.*, **3**, 133-18 (1922).
3. Bota, M., Molnar, A. and Varga, C., On eke land's variational principle in *b*-metric spaces, *Fixed Point Theory*, **12**, no. 2, 21-28 (2011).
4. Boriceanu, M., Fixed point theory for multivalued generalized contraction on a set with two b-metric, studia, univ Babes, *Bolya: Math*, Liv (**3**), 1-14 (2009).
5. Czerwik, S., Non-linear set valued contraction mappings in *b*-metric spaces. *Atti sem math fiq Univ. Modena*, **46**(2), 263-276 (1998).
6. Kir, Mehmet, Kiziltune, Hukmi, on some well known fixed point theorems in *b*-metric space, *Turkish journal of analysis and number theory*, Vol. **1**, no. 1, 13-16 (2013).
7. Pacurer, M., Sequences of almost contractions and fixed points in *b*-metric spaces, Analele University at. de Vest, Timisoara, *Seria Mathematics, Informatics*, **XLVIII**, no. 3, 125-137 (2010).

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