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INTEGRAL SOLUTIONS OF THE HOMOGENEOUS TRINITY QUADRATIC EQUATION $3x^2 + y^2 = 16z^2$

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The homogeneous trinity quadratic equation $3x^2 + y^2 = 16z^2$ is scrutinized for its non-zero unique integral solutions. Integral solutions are derived in four different patterns. A few intriguing relationships between the solutions and a few unique polygonal integers are shown.

KEYWORDS : *Trinity Quadratic equation, Integral Solutions, polygonal number, homogeneous.*

INTRODUCTION

There is a wide range of ternary quadratic equations, see [1-3 & 8-11] for a more thorough understanding. [4-7] has been investigated for the ternary quadratic Diophantine equations of non-trivial integral solutions. In this communication, consider another intriguing ternary quadratic problem $3x^2 + y^2 = 16z^2$ and find various non-trivial integral solution patterns. Additionally, several intriguing connections between the solutions and unique polygonal, centered, Gnomonic star numbers are shown.

NOTATIONS

$$T_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right] \text{ (Polygonal number)}$$

$$\Pr_n = n(n+1) \text{ (Pronic number)}$$

$$Gno_n = 2n-1 \text{ (Gnomonic number)}$$

$$Star_n = 6n(n-1)+1 \text{ (Star number)}$$

$$CH_n = 3n^2 - 3n + 1 \text{ (Centered hexagonal number)}$$

The quadratic homogeneous trinity equation is to be solved for a non-zero integral solution using the following method:

$$3x^2 + y^2 = 16z^2 \quad \text{... (1)}$$

The linear transformation is used in place of

$$x = X + T, y = X - 3T \quad \text{... (2)}$$

In (1) leads to $X^2 + 3T^2 = 4Z^2$ ... (3)

The various approaches to solving (3) and the resulting range of possible integral solutions to (1) are shown under :

Pattern 1 : Assume $z = z(a,b) = a^2 + 3b^2$ ... (4)

where *a* and *b* are positive integers.

Enter 4 as, $4 = (1+i\sqrt{3})(1-i\sqrt{3})$ ... (5)

Applying the factorization method and substitute (4) and (5) in (3),

$$(X+i\sqrt{3})(X-i\sqrt{3}) = (1+i\sqrt{3})(1-i\sqrt{3})[(a+i\sqrt{3}b)^2(a-i\sqrt{3}b)^2]$$

By equating similar concepts and contrasting real and imagined components,

$$X = a^2 - 3b^2 - 6ab$$
$$T = a^2 - 3b^2 + 2ab$$
$$z = a^2 + 3b^2$$

By using the values of X and T from (2), the corresponding integer solutions of (1) are provided by

$$x = a^2 - 6b^2 - 4ab$$
$$y = -2a^2 + 6b^2 - 12ab$$
$$z = a^2 + 3b^2$$

Thought Provoking Results:

1. $-x(G,G) - y(G,G)$ is a Perfect Square
2. $-x(G,G) - y(G,G) + 2z(G,G)$ is a Nasty Number
3. $-x(G,G) + z(G,G)$ is a first abundant number
4. $-2x(G,G) - 2y(G,G) + z(G,G)$ is an abundant Number
5. $-x(G,1) - y(G,1) \equiv 0 \pmod{16}$
6. $-x(G,1) - y(G,1) + T_{6,G} + 2T_{5,G} - Gno_G \equiv 0 \pmod{8}$
7. $3y(G,1) + z(G,1) - Star_G + 15Gno_G \equiv 0 \pmod{38}$
8. $19x(G,1) + 16y(G,1) + 23z(G,1) - T_{22,G} - 37Gno_G \equiv 185 \pmod{14}$

Pattern 2:

Equation (3) has the following form

$$X^2 - 4Z^2 = -3T^2 \qquad \text{... (6)}$$

In the ratio form, write (6) as

$$\frac{X+2z}{-3T} = \frac{T}{X-2z} = \frac{a}{b} \qquad \text{... (7)}$$

The following two equations represent this as its equivalent

$$bX + 3aT + 2bz = 0$$
$$aX - bT - 2az = 0$$

With regard to using the cross ratio multiplication approach,

$$X = -6a^2 + 2b^2$$
$$T = 4ab$$
$$z = -3a^2 - b^2$$

Given (2), the integral solutions to (1) are presented by

$$x = -6a^2 + 2b^2 + 4ab$$
$$y = -6a^2 + 2b^2 - 12ab$$
$$z = -3a^2 - b^2$$

Thought Provoking Results:

1. $x(G,G) - 2y(G,G) + z(G,G)$ is a Perfect square number
2. $x(G,G) - y(G,G) - 2z(G,G)$ is a Nasty Number
3. $x(G,G) - 2y(G,G) - 6z(G,G)$ is a Tetranacci Number and also it gives sum of first six triangular numbers
4. $x(G,1) - y(G,1) \equiv 0 \pmod{16}$

5. $x(G,1)-y(G,1)-z(G,1)-T_{8,G}\equiv 1(\mathrm{mod}\,18)$
6. $-x(G,1)-y(G,1)-4CH_G-10Gno_G\equiv 0\,(\mathrm{mod}\,2)$
7. $-2y(G,1)+2z(G,1)-Star_G-15Gno_G\equiv 0\,(\mathrm{mod}\,8)$
8. $-8x(G,1)-5y(G,1)+10z(G,1)-4T_{26,G}-36Gno_G\equiv 0(\mathrm{mod}\,72)$

Pattern 3:

Equation (3) can be expressed as

$$X^2=4Z^2-3T^2 \qquad .... (8)$$

Let $X(a,b)=4a^2-3b^2$ ... (9)

Applying the factorization approach and substituting (9) into (8) results in

$$(2a+\sqrt{3}b)^2(2a-\sqrt{3}b)^2=(2z+\sqrt{3}T)(2z-\sqrt{3}T)$$

Putting factors that are rational and those that are not rational on an equitable basis,

$$z(a,b)=\frac{1}{2}(4a^2+3b^2)$$

$$T(a,b)=4ab$$

Since we are looking for integer solutions, we need to change *a* and *b* to equal $2G$ and $2J$, respectively.

$$X(2G,2J)=16G^2-12J^2$$

$$T(2G,2J)=16GJ$$

$$z(2G,2J)=8G^2+6J^2$$

Consequently, in context of (2), the integral values to (1) are provided by

$$x=16G^2-12J^2+16GJ$$

$$y=16G^2-12J^2-48GJ$$

$$z=8G^2+6J^2$$

Thought Provoking Results:

1. $x(G,G)-y(G,G)$ is a Perfect square number
2. $x(G,G)-y(G,G)+z(G,G)$ is an abundant number
3. $-y(G,G)+z(G,G)$ is a Smith number
4. $x(G,1)-y(G,1)\equiv 0\,(\mathrm{mod}\,64)$
5. $x(G,1)-y(G,1)+z(G,1)-T_{18,G}\equiv 6\,(\mathrm{mod}\,71)$
6. $8x(G,1)-5y(G,1)+10z(G,1)-16T_{18,G}\equiv 24\,(\mathrm{mod}\,224)$
7. $-8x(1,J)-5y(1,J)-10z(1,J)-16Star_J\equiv 272\,(\mathrm{mod}\,464)$
8. $-y(G,1)-2z(G,1)+CS_G\equiv 16\,(\mathrm{mod}\,16)$

Pattern 4:

The trinity quadratic equation (3) can be expressed as

$$4Z^2-3T^2=X^2*1 \qquad ... (10)$$

Assume $X=4a^2-3b^2$ ... (11)

Write $1=(2+\sqrt{3})(2-\sqrt{3})$ ... (12)

By replacing (11) and (12) in (10) and using the factorization approach, the following result is obtained,

$$(2z+\sqrt{3}T)(2z-\sqrt{3}T)=(2a+\sqrt{3}b)^2(2a-\sqrt{3}b)^2*(2+\sqrt{3})(2-\sqrt{3})$$

Putting factors that are rational and those that are not rational on an equitable basis

$$z=4a^2+3b^2+6ab$$

$$T=4a^2+3b^2+8ab$$

Therefore, in context of (2), the integral values to (1) are provided by

$$x=8a^2+8ab$$

$$y=-8a^2-12b^2-24ab$$

$$z=4a^2+3b^2+6ab$$

Thought Provoking Results:

1. $-x(A,A)-y(A,A)-z(A,A)$ is a triangular number
2. $-10x(A,A)-10y(A,A)-10z(A,A)$ is a Nasty number
3. $-x(A,1)-y(A,1)\equiv 12 \pmod{16}$
4. $-x(A,1)-y(A,1)+z(A,1)-T_{10,A}\equiv 15 \pmod{25}$
5. $x(A,1)+y(A,1)+z(A,1)-T_{10,A}+7A\equiv 0 \pmod{9}$
6. $-10x(A,1)-5y(A,1)-8z(A,1)+8T_{11,A}+32Gno_A\equiv 0 \pmod{4}$
7. $-8x(A,1)-5y(A,1)-10z(A,1)+8T_{18,A}+22Gno_A\equiv 0 \pmod{8}$
8. $-x(A,1)-2y(A,1)-3z(A,1)+T_{10,A}\equiv 15 \pmod{19}$

CONCLUSION

For the trinity quadratic equation $3x^2+y^2=16z^2$, we have given numerous non-zero unique integral solution patterns. To conclude, one can look for further options for solutions and their respective attributes among the various choices.

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