A SURVEY ON STUDY OF GENERALIZED SPECIAL FUNCTION AND FRACTIONAL CALCULUS WITH THEIR APPLICATION

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The present paper introduces the work done on the study of generalized special functions and fractional calculus with their application in various fields. The main object of this paper is to present generalization of special functions, some recurrence relations, transformation formulas, and integral representation are obtained for these new generalizations which is associated with fractional calculus and integral transform.

KEYWORDS: Special function, Fractional Calculus, integral transform

INTRODUCTION

In 17th century, the oxford professor John Wallis interested in the theory of Gamma function and π formula and also represented the elliptic integrals while using cavalieri's primitive forerunner of the calculus. 19th century is known as Golden Age of Special functions due to its huge applications in the development of both mathematics and physics.

Special function and their applications are inspiring in their scope, variety and depth. Not only use in the pure mathematics and its application field of Statistics, Physics and Engineering but new in field of application such as Economics, Optimization, Environment Science, Biology etc. Special functions are a broad area of physically and mathematically relevant functional equations. The special function brought into existence from certain problems of applied Physics and engineering leading to differential equation which had no solution in terms of well known functions then found some new functions which possessed interesting properties such as orthogonallity, integral transformation, differentiation, convergence etc. were called special functions. The study of special functions begun up with the calculus and is as a result one of the oldest branches of analysis.

Fractional Calculus has been interesting field of research work for various years and is sustaining quickly development. The general notion of integrals and derivatives of arbitrary order is an extension of integration and differentiation from the integral order n of the operator $\frac{d^n}{dx^n}$ to arbitrary order. In the beginning period, the order n was taken to be fraction so the study of this branch is called Fractional Calculus. Initially, Leibnitz was try to extend a derivative of $\frac{d^n}{dx^n}$ integer order n.

In 1819, the first mathematician Lacroix represent a paper on fractional derivative. Starting with $y = x^m$, where m is a positive integer, Lacroix easily developed n^{th} derivative. The usefulness of fractional calculus is solving various problems emerging in mathematics, engineering, physics and other applied sciences have made this branch highly popular.

LITERATURE REVIEW

Fractional Calculus allows integer and derivative of any positive real order. It can be considered as a branch of mathematical analysis which deals with integro-differential operators and equations where the integrals of convolution type and kernels of power-law type exhibited. Since Leibnitz's historical regarding consequences of fractional calculus, several attempts were made over the centuries, to formalize a definition which fulfils the criteria of analyticity of the function.

Mention a derivative of arbitrary order by Lacroix [1]. Letnikov [2] solved certain differential equations by the theory of fractional calculus. Laurent [3] generalized Cauchy integral formula by making use of the generalized product rule of Leibnitz. Hardy [4] investigated the properties of integrals of fractional order and also proved the theorems of continuity and summability, seeking analogies to properties which were valid integrals of integer order. Erdelyi and sneddon [5] obtained the solution of dual integral equations using fractional integration. McCollum and Brown [6] made a list of Laplace and inverse Laplace transforms of various fractional operators.

Saxena, Mathai and Haubold [7] studied fractional kinetic equations and obtained some results involving the Mittag-Leffler function. Debnath [8] presented several applications of fractional calculus in various field as diverse as control theory, electric circuits, visco-elasticity and electro-magnetism. Jain and Pathan [9] obtained some relations between integral transforms and fractional calculus operators Wely type. Saxena, Yadav, Purohit and Kalla [10] obtained fractional q integral operator of the basic analogue of the H- function. Mathai and Haubold [11] gave application of pathway models in super statistics, trellis statistics and generalized measure of entropy. Kiryakova [12] authored a review article history of the operators of the generalized fractional calculus and showed that all known fractional integral and differential operators in various areas of analysis happen to fall in the scheme of the generalized fractional calculus. Ozergin, Ozarslan and Altin [13] presented extension of gamma, beta and hyper geometric functions. Agarwal and Chand [14] gave new finite integrals involving product of H –function and Srivastava's polynomial. Luo and Raina [15] extended earlier results to generalized hyper geometric functions and their applications. Baleanu and Agrawal [16] defined generalized fractional integral operators involving the generalized Gauss Hyper geometric functions.

Agarwal, Chand and Karimov [17] obtained certain image formulas of generalized hyper geometric functions. Kumar [18] established a general theorem connecting with Laplace Transform and generalized Weyl Fractional integral operator. Mohammed [19] et. al. presented generalization and extension of beta function with three and four parameters. Rahman [20] *et. al.* introduced a further extension of extended (p, q)-beta function by considering two Mittag-Leffler functions in the kernel. Min Cai Changpin Li [21] is found that fractional calculus has main two different characteristics singularity and non locality from its origin.

MATERIAL AND METHODS

We will use some special functions, fractional calculus operators and integral transform in our proposed research work which is following:

Riemann- Liouville left- sided fractional integral of order α

$$ {}_aI_x^{\ \alpha}\ f(x) = {}_aD_x^{-\alpha}\ f(s) = I_+^{\alpha}\ f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}}, \quad x > a \qquad \text{... (1)}$$

Laplace transform

The Laplace transform of a function *f(t)*, defined for all real numbers $t \geq 0$, is the function *f(s)*, defined by:

$$ f(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \qquad \text{... (2)}$$

The parameter *s* is a complex number $s = \sigma + i\omega$ with real numbers σ and ω.

Mittag –Leffler Functions

Mumtaz Ahmad Khan and Shakeel Ahmed present a new generalization of Mittag-Leffler function in 2013, as

$$ E_{\alpha,\beta,\upsilon,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(Z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}\ (\gamma)_{qn}}{\Gamma(\alpha n+\beta)\,(\upsilon)_{\sigma n}\,(\delta)_{pn}} Z^n, \qquad \text{... (3)}$$

where $\mu, \rho, \gamma, \delta, \alpha, \beta, \upsilon, \sigma, \delta \in C$; $p, q > 0$ and $q \leq Re(\alpha) + p$
min [Re(α), Re(β), Re(υ), Re(ρ), Re(σ), Re(δ), Re(μ), Re(γ) > 0].

CONCLUSION

We shall establish results by using transform methods applying different transforms like Laplace transform, Mellin Transform, Hankel transform etc. to solve certain integrals and derivatives and also find existing results as their special cases. To generalized the functions of fractional calculus and obtain the relations that exist among these functions and fractional calculus operators.

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