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BOUNDEDNESS OF PSEUDO-DIFFERENTIAL OPERATOR ASSOCIATED WITH THE BESSEL TYPE OPERATOR

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In this paper, the pseudo differential operator T_σ associated with the Bessel type operator is defined on the space of even Schwartz functions and obtain a singular integral representation of T_σ. Further we proved that the kernel K_σ satisfies the condition of singular integral theorem. Finally, it is shown that the pseudo differential operator is bounded from $L_{p,a,b}$ into itself for $1 < p < \infty$, when the symbol σ belongs to the class $S^0_{a,b}$.

KEYWORDS : Fourier Bessel transform, Bessel operators, Singular integral operators, generalized translation operators, pseudo differential operators.

INTRODUCTION

The theory of pseudo differential type operators has importance in harmonic analysis and wavelet analysis. The boundedness properties of the pseudo differential operator in the classical harmonic analysis are related to different class of symbols. Many results were obtained by using different methods (see [1], [2], [3], [4], [5]) and many applications were extended to harmonic analysis associated with Bessel type operator (see [6], [7], [8], [9], [10]).

In this paper, the pseudo differential operator T_σ is defined on the space of even Schwartz functions by

$$T_\sigma(f)(x) = \int_0^\infty \sigma(x,\ t) F_{a,b}(f)(t)\ j_{\frac{a-b-a}{2}}(xt)\ d\mu_{a,b}(t)\,,\ x \geq 0 \qquad \dots (1)$$

where $\Phi_{a,b}(f)$ is the Fourier-Bessel type transform of f, j_λ is the normalized Bessel type function of the first kind with order λ and σ a C^∞ complex valued function of $R \times R$. We say that σ belongs to the class of symbols $S^0_{a,\ b}$ if σ is even for each variable and satisfies the following condition:

$$(1+x)^{a-b} |\ \partial_t^r \partial_x^a \sigma\ (x,\ t)| \ \leq\ C_{r,\ s}\ (1+t)^{-r},\ r,\ s\ \in\ N,\ x,\ t\ \geq\ 0, \qquad \dots (2)$$

where $C_{r,\ s}$ is a constant depending only on r and s. We show that T_σ is a singular integral operator given by

$$T_\sigma(f)(x) = \int_0^\infty K_{a,\ b}(x,\ y) f(y)\ d\mu_{a,\ b}(y)\,,\ x\ \geq 0,\ x \notin\ Supp\ (f),$$

where the kernel $K_{a,\,b}$ is singular near $x = y$. We have shown that T_σ satisfies conditions of singular integral theorem on $(0, \infty)$. As a consequence, we obtain that T_σ can be extended to a bounded operator from $L_{p,a,b}$ into itself for $1 < p < \infty$.

Throughout this paper, C denotes a suitable positive constant not necessarily the same in each occurrence.

We denote by $D_e(R)$, the space of even C^∞ -function on R with compact support and $S_e(R)$, the space of even Schwartz functions on R.

PRELIMINARY RESULTS AND NOTATIONS

Consider the Bessel type operator $B_{a,b}$ on$(0, \infty)$ defined by

$$B_{a,\,b} = D_x^2 + \frac{a-b}{x} D_x = \frac{1}{x^{a-b}} D_x (x^{a-b} D_x),\ D_x \equiv \frac{d}{dx}, \quad \text{... (3)}$$

for a real parameter $(a - b) > 0$.

The following initial value problem has a unique solution $j_{\frac{a-b-1}{2}}(\lambda)$ (see [6]):

$$B_{a,b}(f\,)(x) = -\lambda^2 f\,(x), f(0) = 1, f'(0) = 0,$$

where $\lambda \in C$.

Let $\mu_{a,\,b}$ be the weighted Lebesgue measure on $[0, \infty)$ given by

$$d\mu_{a,\,b}(x) = \frac{x^{a-b}}{2^{\frac{a-b-1}{1}}\,\Gamma\left(\frac{a-b+1}{2}\right)}$$

We denote $L_{p,a,b}$ the space $L^p(R^+, d\mu_{a,b}$ and we use $\|.\|_{p,a,b}$ as a shorthand for $\|.\|_{L_{p,a,b}}$for every $1 \le p < \infty$.

The Bessel-Fourier type transform is defined for $f \in L_{1,a,b}$ by

$$F_{a,\,b}(f)(x) = \int_0^\infty f(t)\ j_{\frac{a-b-1}{2}}(x,\ t)\ d\mu_{a,\,b}(t)\,,\ x\ \in\ (0,\ \infty)$$

For all $x, y, z \in (0, \infty)$, let

$$W_{a,\,b}(x,\ y,\ z) = \begin{cases} d_{a,\,b}\, \dfrac{\{[(x+y)^2 - z^2]\ [z^2 - (x-y)^2]\}^{\frac{a-b-2}{2}}}{(xyz)^{a-b-1}} & \text{if } |x-y| < z < x+y, \\ 0, & \text{otherwise} \end{cases}$$

where
$$d_{a,\,b} = \frac{2^{\frac{3-a+b}{2}}\left(\Gamma\left(\frac{a-b+1}{2}\right)\right)^2}{\sqrt{\pi}\ \Gamma\left(\frac{a-b+1}{2}\right)}$$

From [6], we have the following product formula:

$$\int_0^\infty W_{a,\,b}(x,\ y,\ t)\ j_{\frac{a-b-1}{2}}(xz)\ d\mu_{a,\,b}(t) = j_{\frac{a-b-1}{2}}(xz)\ j_{\frac{a-b-1}{2}}(yz),\ x, y > 0,\ z \ge 0.$$

and $W_{a,b}$ is such that

$$\int_0^\infty W_{a,\,b}(x,\ y,\ t)\ d\mu_{a,\,b}(z) = 1 \quad \text{... (4)}$$

Now the generalized translation operator associated with the Bessel type operator for a continuous function f on $[0, \infty)$ is defined as

$$\tau_x(f)\ (y) = c_{a,\,b} \int_0^{\pi} f(\Delta\ (x,\ y,\ \theta)\ (sin\ \ \theta)^{a-b-1} d\theta,$$

where $\Delta(x,\ y,\ \theta) = \sqrt{x^2 + y^2 - 2xy\ cos\theta} =\ |x - ye^{i\theta}|$ and $c_{ab} = c_{\frac{a-b-1}{2}} = \frac{\Gamma\left(\frac{a-b+1}{2}\right)}{\sqrt{\pi}\ \Gamma\left(\frac{a-b}{2}\right)}$.

Now we have the following properties from [6]

(1) For a continuous function f on $[0, \infty)$,

$$\tau_x(f)\ (y) = \int_0^{\infty} f(z)\ W_{a,\,b}(x, y,\ z)\ d\mu_{a,\,b}(z)\,,\ x,\ y > 0 \qquad ...\ (5)$$

(2) $$F_{a,\,b}(\tau_x(f))\ (y) = j_{\frac{a-b-1}{2}}(xy)\ F_{a,\,b}(f)\ (y) \qquad ...\ (6)$$

for all $f \in L_{1,a,b},\ x, y\ \geq\ 0.$

Now we recall fundamental singular integral theorem from ([5], chap.1).

Theorem 2.1. Let K be a measurable function on $\{(x, y), x \geq 0, y \geq 0, x \neq y\}$ and T be a bounded operator from $L_{2,a,b}$ into itself such that

$$T\ (f)\ (x) = \int_0^{\infty} K\ (x,\ y)\ f(y)\ d\mu_{a,\,b}(y), x \geq 0, \qquad ...\ (7)$$

for any compactly supported f in $L_{2,a,b}$ and all $x \notin$ Supp(f). If K satisfies

$$\int_{|x-y|\ >\ 2\delta} |K\ (x,\ y)\ - K\ (x,\ y')| d\mu_{a,\,b}(x) \leq\ C, \qquad ...\ (8)$$

for all $\delta > 0$ and $y, y' \in [0, \infty)$ with $|y - y'| \leq \delta$, then T can be extended to bounded operator from $L_{p,a,b}$ into itself for $1 < p \leq 2$.

SINGULAR INTEGRAL REPRESENTATION OF T_σ

In this section we have obtained a singular integral representation of T_σ. Further we have shown that the kernel K_σ satisfies the condition of singular integral theorem.

Lemma 3.1. If $\sigma \in S^0_{a,b}$, then for $m \in \mathbb{N}$, we have

$$(1\ +\ x)^{a-b} |\ B^m_{a,b}\ (\sigma(x,\bullet))\ (\xi)|\ \leq\ c_m(1\ +\ \xi)^{-\ 2m},\ x\ \geq\ 0,\ \xi\ >\ 0, \qquad ...\ (9)$$

where $B^n_{a,\,b} = B_{a,\,b}$ o $B_{a,\,b}$ …. o $B_{a,\,b}$ and c_m is a constant which depends only on m.

Proof. By induction, we know that

$$B_{a,\,b}\ (\sigma,\ (x,\ \bullet))\ (\xi) = \partial^2_\xi \sigma\ (x,\ \xi)\ \int_0^1 \partial^2_\xi \sigma\ (x,\ t\xi)\ dt\,, x \geq 0,\ \xi > 0.$$

By using (2) and above result we have

$$(1 + x)^{a-b} |B^m_{a,\,b}\ (\sigma,\ (x,\ \bullet))\ (\xi)|\ \leq A_m, x \geq 0, \xi > 0 \qquad ...\ (10)$$

where A_m is depends only on m.

On the other side by using induction, we obtain

$$B^m_{a,\,b}\ (\sigma,\ (x,\ \bullet))\ (\xi)\ =\ \xi^{-2m} \sum_{0\ \leq\ i\ \leq\ 2n} a_i \xi^i \partial^i_\xi\ \sigma\ (x,\ \xi), x \geq 0, \xi > 0, \qquad ...\ (11)$$

where a_i is a real constant. Finally, again by using (2), we get

$$\xi^{2n}(1 + x)^{a-b} |B^m_{a,\,b}\ (\sigma,\ (x,\ \bullet))\ (\xi)|\ \leq B_m, x \geq 0, \xi > 0.... \qquad ...\ (12)$$

Here B_m depends only on m. Thus from (10) and (12) we obtain (9).

Lemma 3.2. Let $\sigma \in S^0_{a,\,b}$. Then k_σ is in C^{∞} $([0, \infty) \times [0, \infty))$

and satisfies for all *r*, *s*, *m*∈ℕ.

$$\xi^{2n}(1+x)^{a-b}|B^m_{a,\,b}\ (\sigma,\ (x,\bullet))\ (\xi)|\ \le B_m, x\ge 0, z>0, \ldots \tag{13}$$

where *C* depends only on *r*, *s*, *m*.

Proof. Let η bean element $D_e(R)$such that$\eta\ (\xi)=1$ for $|\xi|<1$ and $\eta(\xi)=0$ for $|\xi|\ge 2$. Now for a function $\delta(\xi)=\eta(\xi)-\eta(2\xi)$, we have obtained the following partition of unity,

$$\eta(\xi)+\sum_{j=1}^{\infty}\delta(z^{-j}\xi)=1,\ \xi\ge\ 0.$$

If $\sigma\in S^0_{a,\,b}$, then we can have

$$\sigma(x,\ \xi)=\ \sum_{j=1}^{\infty}\sigma_j\ (x,\ \xi)\,,\ x,\ \xi\ \ge\ 0,$$

where $\sigma_j(x,\xi)\ =\ \sigma(x,\xi)\,\delta(z^{-j}\xi),\ \sigma_0(x,\xi)=\sigma\,(x,\xi)\,\eta(\xi)$.

Set $k_j(x,z)\ =\ \int_0^{\infty}\sigma_j(x,\ \xi)\ j_{\frac{a-b-1}{2}}(z\xi)\ d\mu_{a,\,b}(\xi)$, for all $x,z\ge 0$, since $\sigma_j(x,\bullet)$ has a compact support.

Now we can obtain

$$\partial_z^x\partial_x^s k_j(x,\ z)=\int_0^{\infty}\xi^r\partial_x^s\sigma_j(x,\ \xi)\ j_{\frac{a-b-1}{2}}(z\xi)\ d\mu_{a,\,b}(\xi).$$

Integration by parts gives us

$$z^r\partial_z^r\partial_x^s k_j(x,\ z)=\int_0^{\infty}w_j(x,\ \xi)\ j_{\frac{a-b-1}{2}}(z\xi)\ d\mu_{a,\,b}(\xi)\,,\ x\ge 0,\ z>0,$$

where for $x\ge 0$, $\xi>0$,

$$w_j(x,\ \xi)=\xi^{-(a-b)}\partial_\xi^r|\ \xi^{(a-b+r)}\partial_x^s\sigma_j(x,\ \xi)]$$

$$=\sum_{k\,=\,0}^{r}(r+a-b)\ \ldots\ (k+a-b+1)\binom{r}{k}\ \xi^k\partial_\xi^k\partial_x^s\sigma_j(x,\ \xi).$$

As w_j is supported in $[z^{j-1},z^{j+1}]$, by using (2), we can obtain

$$|\ \partial_\xi^n w_j\ (x,\ \xi)|\ \le\ C2^{-jm}$$

for all $m\in\mathbb{N}$.

But from (11) it is clear that

$$|\ l^n_{a,\,b}(w_j\ (x,\bullet))|\ \le\ C2^{-2jm},$$

for $m\in\mathbb{N}$.

As $(-1)^m\ z^{r+2m}\partial_z^r\partial_x^s k_j(x,\ z)=\int_0^{\infty}B^m_{a,\,b}(w_j\ (x,\bullet))\ (\xi)j_{\frac{a-b-1}{2}}(z\xi)\ d\mu_{a,\,b}(\xi)$,

we can obtain

$$|z^{r+2m}\partial_z^r\partial_x^s k_j(x,\ z)|\le C2^{(a-b+1-2m)\ j}, \tag{14}$$

where *C* is a constant independent of *j*.

Now choose *m* such that $m>\frac{a-b+1}{2}$. By using (14) we conclude that $\sum_{j=0}^{\infty}k_j$is a C^{∞}-function on $[0,\infty)\times[0,\infty)$. Thus, we have

$$(-1)^m z^{2m}=\ \sum_{j=0}^{\infty}\int_0^{\infty}B^m_{a,\,b}(\sigma_j(x,\bullet))\ (\xi)\ j_{\frac{a-b-1}{2}}(\xi z)\ d\mu_{a,\,b}(\xi)$$

$$= \int_0^\infty B_{a,\,b}^m(\sigma_j(x,\,\bullet))\ (\xi)\ j_{\frac{a-b-1}{2}}(\xi z)\ d\mu_{a,\,b}(\xi)$$
$$= (-1)^m z^{2m} k_\sigma(x,\ z).$$

We can obtain that

$$k_\sigma(x,\ z) = \sum_{j=0}^\infty k_j(x,\ z)$$

and $\partial_z^r \partial_x^s k_\sigma(x,\ z) = \sum_{j=0}^\infty \partial_z^r \partial_x^s k_j(x, z)$, for $(x,\ z) \in [0,\ \infty) \times\ [0,\ \infty]$.

Now to prove (13), it is sufficient to estimate the last sum. Consider the case $0 < z \leq 1$.

We can write

$$\sum_{j=0}^\infty |\partial_z^r \partial_x^s k_j(x,\ z)| = \sum_{2^j\ \leq\ z^{-1}}^\infty |\partial_z^r \partial_x^s k_j(x,\ z)| + \sum_{2i\ >\ z^{-1}}^\infty |\partial_z^r \partial_x^s k_j(x,\ z)|.$$

Now using (14) with $n > \frac{a-b+1}{2}$, the second sum is dominated by

$$Cz^{-2m-r} \sum_{2^j\ \leq\ z^{-1}}^\infty 2^{(a-b+1-2m)\ j} \leq Cz^{-(a-b+1)-r} \leq Cz^{-(a-b+1)-r-m}.$$

For the case $z > 1$ and $m \in \mathbb{N}$, we take $m > \frac{a-b+1}{2} + \frac{n}{2}$ in (14), so we have

$$\sum_{j\ =0}^\infty |\partial_z^r \partial_x^s k_j(x,\ z)| \leq Cz^{-r-2m} \sum_{j\ =0}^\infty 2^{(a-b+1-2m)\ j} \leq Cz^{-r-2m} \leq Cz^{-(a-b+1)-r-m}.$$

Theorem 3.3. Let $\sigma \in S_{a,\,b}^0$, then there exists a continuous function k_σ on $[0, \infty) \times [0, \infty)$ such that for $m \in \mathbb{N}, m > \frac{a-b+1}{2}$,

$$|k_\sigma(x,\ z)|\ \leq\ \frac{c_m}{z^{2m}},\quad x \geq\ 0,\ z > 0, \qquad \text{... (15)}$$

and we have

$$T_\sigma(f)\ (x) = \int_0^\infty K_\sigma(x,\ y)\ f(y)\ d\mu_{a,\,b}(y), \qquad \text{... (16)}$$

for $f \in S_e(R)$ and $x \notin$ Supp(f) where K_σ is a kernel given on $\{x \geq 0, y \geq 0, x \neq y\}$ by

$$K_\sigma(x,\ y) = c_{a,\,b} \int_0^\pi k_\sigma(x,\ \Delta\ (x,\ y,\ \theta))\ (sin\ \ \theta)^{a-b-1}. \qquad \text{... (17)}$$

Here c_m is constant which depends only on m.

Proof. Put $\rho_m(\xi)\ =\ (-1)^m \xi^{2m},\ m \in \mathbb{N}$.

Let $f \in S_e(R), x\ \geq\ 0$ and $x \notin Supp\ (f)$. Assume that the complement of its support is non empty. It is clear that $\underline{0 \notin Supp\ (\tau_x(f))}$. We know that

$$F_{a,\,b}(\tau_x(f))\ (\xi) = B_{a,\,b}^m \left(F_{a,\,b}\left(\frac{\tau_x(f)}{\rho_m}\right)\right)\ (\xi).$$

We can write by using (1) and (6)

$$T_\sigma(f)\ (x) = \int_0^\infty \sigma\ (x,\ \xi)\ F_{a,\,b}\ \big(\tau_x(f)\big)\ (\xi)\ d\mu_{a,\,b}(\xi)$$
$$= \int_0^\infty \sigma\ (x,\ \xi)\ B_{a,\,b}^m\ (F_{a,\,b}\left(F_{a,\,b}\left(\frac{\tau_x(f)}{\rho_m}\right)\right)\ (\xi)\ d\mu_{a,\,b}(\xi)$$

Now by integration by parts and using relation (3), we have

$$T_\sigma(f)\ (x) = \int_0^\infty B_{a,\,b}^m\ (\sigma\ (x,\bullet))\ F_{a,\,b}\ \left(\frac{\tau_x(f)}{\rho_m}\right)\ (\xi)\ d\mu_{a,\,b}(\xi)$$

Let $m > \frac{a-b+1}{2}$. Then from (9), it is clear that $B_{a,\,b}^m(\sigma\ (x,\bullet)) \in L_{1,\ a,\ b}$

and $$T_\sigma(f)\ (x) = \int_0^\infty F_{a,\,b}(B^m_{a,\,b}\ (\sigma\ (x,\ \bullet))\ (z)\ \left(\frac{\tau_x(f)\ (z)}{\rho_m(z)}\right)\ d\mu_{a,\,b}(z)$$

Set $$k_\sigma(x,\ z) = \frac{F_{a,\,b}(B^m_{a,\,b}\ (\sigma\ (x,\ \bullet))\ (z)}{\rho_m(x)}$$
$$= \frac{(-1)^m}{z^{2m}} \int_0^\infty B^m_{a,\,b}\ (\sigma\ (x,\ \bullet))\ (\xi)\ j_{\frac{a-b-1}{2}}(\xi z)\ d\mu_{a,\,b}(z).$$

Now by using (9), we can obtain (15).

Thus, for $x \geq 0$ and $x \notin$ Supp(f), we can have
$$T_\sigma(f)\ (x) = \int_0^\infty k_\sigma(x,\ z)\ \tau_x(f)\ d\mu_{a,\,b}(z)$$

Hence by the Fubini's theorem and a change of variable $z = \Delta(x, y, \theta)$ and by using (5), (15), we obtain (16). We can also have
$$T_\sigma(x,\ y) = \int_0^\infty k_\sigma\ (x,\ z)\ W_{a,\,b}\ (x,\ y,\ z)\ d\mu_{a,\,b}(z), \qquad x,\ y > 0,\ x \neq y$$

Theorem 3.4. There exists constants $A > 0$ and $A' > 0$ such that for all $\delta > 0$ and $y, y' \geq 0$ with $|\, y - y'\,| \leq \delta$, we have
$$\int_{|\,x\,-\,y\,|\,>\,2\delta} |K_\sigma\ (x,\ y) - K_\sigma(x,\ y')| d\mu_{a,\,b}(x) \leq A \qquad \text{... (18)}$$
and
$$\int_{|\,x\,-\,y\,|\,>\,2\delta} |K_\sigma\ (y,\ x) - K_\sigma(y',\ x)| d\mu_{a,\,b}(x) \leq A', \qquad \text{... (19)}$$
where K_σ is given by (17).

Proof. Suppose that $x, y, y' \geq 0$ and $\delta > 0$ such that $|\, y - y'\,| \leq \delta$ and $|\, x - y\,| > 2\delta$, we get (17).

Now $K_\sigma(x,\ y)\ - K_\sigma(x,\ y')$
$$= c_{a,\,b}(y - y') \int_0^\pi \int_0^1 \partial_y \Delta\ (x,\ y_t,\ \theta)\ \partial_z\ k_\sigma\ (x,\ \Delta\ (x,\ y_t,\ \theta))\, sin^{(a-b-1)}\,\theta\, d\theta\, dt$$
where $y_t\ = y' +\ t\ (y\ -\ y')$. But as
$$|\partial_y \Delta\ (x,\ y_t,\ \theta)|\ = \frac{|\ y_t - x\cos\ \theta|}{\sqrt{(y_t - x\cos\theta)^2 + (x\sin\theta)^2}} \leq 1,$$
by using (13) with $n = 0$, we obtain
$$K_\sigma(x,\ y)\ - K_\sigma(x,\ y') = C|y - y'|\ \int_0^\pi \int_0^1 |\partial_y k_\sigma(x,\ \Delta\ (x,\ y_t,\ \theta)|\ \ sin^{(a-b-1)}\,\theta\ \ d\theta\ \ dt$$
$$= C\delta\ \int_0^1 \int_0^\infty |\partial_y k_\sigma(x,\ z)|\ W_{a,\,b}(x,\ y_t,\ z) d\mu_{a,\,b}\ (z)\, dt$$
$$= C\delta\ \int_0^1 \int_0^\infty \frac{1}{z^{a-b+2}} W_{a,\,b}(x,\ y_t,\ z) d\mu_{a,\,b}\ (z)\ \ dt$$

By using (4) and Fubini's theorem, we get
$$\int_{|x\,-\,y|\,>\,2\delta} |K_\sigma(x,\ y)\ - K_\sigma(x,\ y')|\ d\mu_{a,\,b}(x)$$
$$\leq\ \int_0^1 \int_0^\infty \int_{|x\,-\,y|\,>\,2\delta} \frac{1}{z^{a-b+2}} W_{a,\,b}(x,\ y_t,\ z) d\mu_{a,\,b}\ (x)\ \ d\mu_{a,\,b}(z)\ \ dt$$
$$\leq\ C\delta \int_0^\infty \frac{1}{z^2} dz\ \ \leq A.$$

This shows that (18) is proved.

Further, using the process as above, we have

$$K_\sigma(y,\ z)\ - K_\sigma(y',\ x) = c_{a,\ b}(y-y')\ \int_0^\pi \int_0^1 G\ (x,\ y_t,\ \theta)|\ sin^{a-b-1}\theta\ \ d\theta,$$

where $G(x,\ y_t,\ \theta) = \partial_x k_\sigma(y_t,\ \Delta\ (x, y_t, \theta)) + \partial_x \Delta\ (x, y_t, \theta)\ \partial_z k_\sigma(y_t,\ \Delta\ (x, y_t, \theta))$.

Once again by using (13) with $n = 1$, we obtain

$$K_\sigma(y,\ z)\ - K_\sigma(y',\ x) \leq\ |y-y'|\ \int_0^1 \int_0^\infty |\partial_x k_\sigma(y_t,\ z)| + |\partial_z k_\sigma\ (y_t,\ z)|)\ W_{a,\ b}\ (x,\ y_t,\ z)\ d\mu_{a,\ b}(z)dt$$

$$\leq C\delta\ \int_0^1 \int_0^\infty \frac{1}{z^{a-b+2}} W_{a,\ b}\ (x,\ y_t,\ z)\ d\mu_{a,\ b}(z)\,dt.$$

Now we get

$$\int_{|x\ -\ y|\ >\ 2\delta} \left|K_\sigma(y,\ x)\ - K_\sigma(y',\ x)\right|\ d\mu_{a,\ b}(x)$$

$$\leq\ C\delta\ \int_0^1 \int_0^\infty \int_0^\infty \frac{1}{z^{a-b+2}} W_{a,\ b}(x,\ y_t,\ z)\ d\mu_{a,\ b}(x)\,d\mu_{a,\ b}(z)\,dt$$

$$\leq A'$$

which proves (19).

$\mathscr{L}^P$-BOUNDEDNESS OF THE OPERATOR T_σ

In this section we need the following lemmas.

Lemma 4.1. Let $\sigma \in S^0_{a,\ b}$, If σ has compact x-support then there exists a constant $C > 0$ such that, for all $f \in S_e(R)$

$$\|T_\sigma(f)\|_{2,a,b} \leq C\|f\|_{2,a,b}.$$

Proof. Let

$$\rho\ (\lambda,\ \xi) = \int_0^\infty \sigma(x,\ \xi)\ j_{\frac{a-b-1}{2}}\ (x\lambda)\ d\mu_{a,b}(x),\qquad \lambda, \xi\ \geq 0,$$

by using inversion formula [6], we have

$$\sigma(\lambda,\ \xi) = \int_0^\infty \rho(\lambda,\ \xi)\ j_{\frac{a-b-1}{2}}\ (x\lambda)\ d\mu_{a,b}(\lambda),\quad x, \xi\ \geq 0,$$

Integrating by parts we get

$$(-1)^m \lambda^{2m}\ \rho\ (\lambda,\ \xi) = \int_0^\infty B^m_{a,\ b}\ (\rho\ (\bullet,\ \xi))\ (x)\ j_{\frac{a-b-1}{2}}\ (x\lambda)\ d\mu_{a,b}(x);$$

for each $m \in \mathbb{N}$.

It can be noted that $B^m_{a,\ b}(\sigma(\bullet,\ \xi))(x)$ is bounded uniformly in ξ and has compact x-support, then for $m \in \mathbb{N}$, $\lambda \to \lambda^{2m}\,|\ \rho(\lambda,\ \xi)|$ is bounded uniformly in ξ and we have

$$\sup_{\xi \geq\ 0} |\rho\ (\lambda,\ \xi)|\ \leq\ \frac{c_m}{(1+\lambda^2)^m},\ \lambda\ \geq 0. \qquad \text{... (20)}$$

where c_m is constant independent of ξ. We can choose $m > \frac{a-b+1}{2}$ and then by using (20), we have $\rho\ (\bullet,\ \xi) \in L_{1,\ a,\ b}$ and by Fubini's theorem, we can write for $f \in S_e$ (R) and $x \geq 0$

$$T_\sigma(f)\ (x) = \int_0^\infty \sigma\ (x,\ \xi)\ F_{a,\ b}(\xi)\ j_{\frac{a-b-1}{2}}(x\xi)\ d\mu_{a,\ b}(\xi)$$
$$= \int_0^\infty T^\lambda(f)\ (x)\ d\mu_{a,\ b}(\lambda),$$

where $$T^\lambda(f)\ (x) = j_{\frac{a-b-1}{2}}(x\lambda)\ T_{\rho\ (\lambda,\ \bullet)}(f)\ (x),$$

with $$T_{\rho\ (\lambda,\ \bullet)}(f)\ (x) = \int_0^\infty \rho\ (\lambda,\ \xi)\ F_{a,\ b}(f)\ (\xi)\ j_{\frac{a-b-1}{2}}(x\xi)\ d\mu_{a,\ b}(\xi).$$

By using (20) and Plancherel formula from [6], we have

$$\left\|T^\lambda(f)\right\|_{2,a,b} \le \left\|T_{\rho\ (\lambda,\ \bullet)}(f)\right\|_{2,a,b} = \left\|\rho\ (\lambda,\ \bullet)\ F_{a,\ b}(f)\right\|_{2,a,b}$$
$$\le \frac{c_m}{(1+\lambda^2)^m}\|f\|_{2,a,b}.$$

Now we can write

$$\int_0^\infty |T_\sigma(f)\ (x)|^2 d\mu_{a,\ b}(x)$$
$$\le \int_0^\infty \int_0^\infty \left(\int_0^\infty |T^{\lambda_1}(f)(x)||\ T^{\lambda_2}(f)(x)|\ d\mu_{a,\ b}(x)\right) d\mu_{a,b}(\lambda_1)\ \ d\mu_{a,\ b}(\lambda_2).$$

Now by using Schwartz inequality, we get

$$\int_0^\infty |T^{\lambda_1}(f)\ (x)||T^{\lambda_2}(f)(x)|\ d\mu_{a,\ b}(x) \le c_m^2\ \frac{1}{(1+\lambda_1^2)^m\ (1+\lambda_2^2)^m}\,||f||^2_{2,\ a,\ b}\,.$$

Thus, we obtain

$$||T_\sigma(f)||_{2,\ a,\ b} \le C_m||f||_{2,\ a,\ b} \int_1^\infty \frac{1}{(1+\lambda_1^2)^m}\, d\mu_{a,\ b}(\lambda) \le\ C||f||_{2,\ a,\ b}\,.$$

Thus, result is proved.

Lemma 4.2. Let $\theta \in D_e(R)$ be supported in $[-2, 2]$, $x_0 > 3$ and η the function defined on R by $\eta\ (x)\ =\ \theta\ (x\ +\ x_0) + \theta(x - x_0)$, then for $\sigma \in B^0_{a,\ b}$, we have

$$||T_{\eta\sigma}(f)||_{2,\ a,\ b}\ \le\ C||f||_{2,\ a,\ b},\quad f \in\ S_e(R) \qquad \text{... (21)}$$

where C is a constant independent of x_0 and $\eta\sigma$ denote the symbol in $S^0_{a,\ b}$, defined by

$$\eta\sigma\ (x,\xi)\ =\ \eta\ (x)\ \sigma\ (x,\xi).$$

Proof. It is clear that $\eta\sigma$ satisfies equation (2) with $C_{r,s}$ independent of x_0 and has compact x-support $x_0 - 2 \le\ |x|\ \le\ x_0 + 2$, then by Lemma 4.1, there exists a constant C such that

$$||T_{\eta\sigma}(f)||_{2,\ a,\ b}\ \le\ C||f||_{2,\ a,\ b},\quad f \in\ S_e(\mathrm{R}).$$

Now we show that the inequality (20) holds with a constant c_m independent of x_0 in order to confirm that the constant C is independent of x_0.

Notice that $l^m_{a,\ b}(\eta\sigma\ (\bullet,\ \xi))\ (x)$ can be written as sum of terms of the form

$$\frac{\eta^{(r)}(x)\ \partial_x^s\sigma\ (x,\ \xi)}{x},\ r,\ s, \in \mathbb{N}.$$

From (2), $|(1 + x)^{a-b} l^m_{a,\ b}(\eta\sigma)\ (\bullet,\ \xi)\ (x)|$ is uniformly bounded in ξ and x_0 for $x \ge 1$ and $m \in \mathbb{N}$. If we design the Fourier-Bessel transform of $\eta\sigma\ (\bullet, \xi)$ by $\rho(\bullet, \xi)$ then we have for all $m \in \mathbb{N}$

$$|\lambda^{2m}\rho(\lambda,\ \xi)|\ \leq\ \int_{x_0-2}^{x_0+2}|B_{a,\ b}^{m}\ (\eta\sigma\ (\bullet,\ \xi))\ (x)|\ d\mu_{a,b}(x) \leq c_m,$$

λ, ξ 0, where c_m is a constant which is independent of x_0 and ξ. Thus (21) is proved.

Lemma 4.3. Let $\sigma \in S_{a,\ b}^{0}$, then there exist a constant $M > 0$ such that for each $x_0 > 3$ and $f \in S_e(R)$, we have

$$\int_{x_0-1}^{x_0+1}|T_\sigma\ (f)\ (x)\ |^2 d\mu_{a,b}(x) \leq\ M\int_0^\infty \frac{|f(x)|^2}{(1+|x-x_0|)^2}\ d\mu_{a,\ b}(x) \qquad \text{... (22)}$$

Proof. Let $w \in D_e(R)$ such that $w(x) = 1$ in $[-2, 2]$, $0 \leq w(x) \leq 1$ and $\text{Supp}(w) \subset [-3, 3]$.

For $x_0 > 3$, we put $\varphi(x) = w(x + x_0) + w(x - x_0)$.

Now for $f \in S_e(R)$, we can write $f\ =\ \varphi f\ + (1-\varphi)f = f_1 + f_2$, where f_1 supported in $x0 - 3 \leq |x| \leq\ x_0 + 3$, f_2 supported outside $x_0 - 2 \leq |\ x\ | \leq x_0 + 2$ and $|\ f_1|, |f_2| \leq |f|$. Now we can choose $\theta \in D_e\ (R)$ such that $\theta(x) = 1$ in $[-1, 1]$ and $\text{Supp}(\theta) \subset [-2, 2]$, then the function η defined by $\eta(x) = \theta(x + x_0) + \theta(x - x_0)$ is supported in $x_0 - 2 \leq |x| \leq x_0 + 2$ and $\eta(x) = 1$ in $x_0 - 1 \leq |x| \leq x_0 + 1$, so by the Lemma 4.2, we have

$$\int_{x_0-1}^{x_0+1}|T_\sigma\ (f_1)\ (x)\ |^2 d\mu_{a,b}(x) = \int_{x_0-1}^{x_0+1}|T_{\eta\sigma}(f_1)\ (x)|^2\ d\mu_{a,\ b}(x)$$

$$\leq\ \int_0^\infty |T_{\eta\sigma}(f_1)\ (x)|^2\ d\mu_{a,\ b}(x)$$

$$\leq\ C\ \int_0^\infty |f_1\ (x)|^2\ d\mu_{a,\ b}(x)$$

$$\leq\ C\ \int_{x_0-3}^{x_0+3}|f(x)|^2\ d\mu_{a,\ b}(x)$$

$$\leq\ C\int_0^\infty \frac{|f\ (x)|^2}{(1\ +\ |x-x_0|)^2}\ d\mu_{a,\ b}(x)$$

If $x \in [x_0 - 1, x_0 + 1]$, we have $x \notin \text{Supp}\ (f_2)$ and by (16), we can write

$$T_\sigma\ (f_2)\ (x) = \int_0^\infty K_\sigma(x,\ y)\ f_2(y)\ d\mu_{a,\ b}(y)$$

For all $x \in [x_0 - 1, x_0 + 1], y \notin [x_0 - 2, x_0 + 2]$, we have

$$K_\sigma\ (x,\ y) = \int_0^\infty k_\sigma(x,\ z)\ W_{a,\ b}(x,\ y,\ z)\ d\mu_{a,\ b}(z) = \int_0^\infty k_\sigma(x,\ z)\ W_{a,\ b}(x,\ y,\ z)\ d\mu_{a,\ b}(z)$$

By using (15) when $m > \frac{a-b}{2} + 2$, we get

$$|T_\sigma\ (f_2)\ (x)|\ = C\int_0^\infty \int_1^\infty |f_2(y)|\ \frac{W_{a,\ b}(x,\ y,\ z)}{z^{a-b+4}} d\mu_{a,\ b}(y).$$

Now by Schwartz-inequality, we have

$$|T_\sigma\ (f_2)\ (x)|\ = C\int_0^\infty \int_1^\infty |f_2(y)|^2\ \frac{W_{a,\ b}(x,\ y,\ z)}{z^{a-b+4}} d\mu_{a,\ b}(z)\ d\mu_{a,\ b}(y)$$

$$= C\int_0^\infty \int_1^\infty |f_2(y)|^2\ \frac{W_{a,\ b}(x,\ y,\ z)}{(1+\ |y-x_0|)^2 z^{a-b+2}} d\mu_{a,\ b}(z)\ \ d\mu_{a,\ b}(y).$$

Now we can obtain,

$$\int_{x_0-1}^{x_0+1}|T_\sigma\ (f_2)\ (x)\ |^2 d\mu_{a,b}(x) \leq C\int_0^\infty \frac{|f(y)|^2}{(1+\ |y-x_0|)^2}\int_1^\infty \int_{x_0-1}^{x_0+1}\frac{1}{z^{a-b+2}}$$

$$\times\ W_{a,\ b}(x,\ y,\ z)\ d\mu_{a,\ b}(x)\ d\mu_{a,\ b}(z)\ d\mu_{a,\ b}(y).$$

Now we can obtain by using (4),

$$\int_1^\infty \int_{x_0-1}^{x_0+1} \frac{1}{x^{a-b+2}} W_{a,\,b}\ (x,\ y,\ z)\ d\mu_{a,\,b}(x)\, d\mu_{a,\,b}(z)$$
$$\leq \int_1^\infty \int_0^\infty \frac{1}{z^{a-b+2}} W_{a,\,b}\ (x,\ y,\ z)\ d\mu_{a,\,b}(x)\, d\mu_{a,\,b}(z) \leq 1.$$

Then we find

$$\int_{x_0-1}^{x_0+1} |T_\sigma\ (f_2(x))\ |^2 d\mu_{a,b}(x) \leq C \int_0^\infty \frac{|f(y)|^2}{(1+\ |y-x_0|)^2} d\mu_{a,b}(y).$$

Finally, by writing $T_\sigma(f) = T_\sigma(f_1) + T_\sigma(f_2)$, we can prove (22).

Theorem 4.4. If $\sigma \in S^0_{a,\,b}$, then the operator T_σ initially defined on $S_e(R)$ can be extended to a bounded operator $L_{2,a,b}$ into itself.

Proof. We integrate in (22) w.r.t x_0, we obtain

$$\int_1^\infty \int_{x_0-1}^{x_0+1} |T_\sigma(f)\ (x)|^2\ d\mu_{a,\,b}(x)\, dx_0 = \int_2^4 |T_\sigma(f)\ (x)|^2\ (x-2)\ d\mu_{a,\,b}(x)$$
$$+ \int_4^\infty |T_\sigma(f)\ (x)|^2\ d\mu_{a,\,b}(x)$$
$$\geq \int_4^\infty |T_\sigma(f)\ (x)|^2\ d\mu_{a,\,b}(x).$$

On other side,

$$\int_3^\infty \int_0^\infty \frac{|f(y)|^2}{(1+\ |y-x_0|)^2}\ d\mu_{a,\,b}(y) dx_0 \leq \int_{-\infty}^\infty \int_0^\infty \frac{|f(y)|^2}{1+\ (y-x_0)^2}\ d\mu_{a,\,b}(y)\ dx_0$$
$$= \int_{-\infty}^\infty \int_0^\infty \frac{|f(y)|^2}{1+\ t^2}\ d\mu_{a,\,b}(y)\, dt$$
$$= \pi \int_0^\pi |f(y)|^2\ d\mu_{a,\,b}(y).$$

Thus, we obtain

$$\int_4^\infty |T_\sigma(f)(x)|^2\ d\mu_{a,\,b}(x) \leq CC\|f\|^2_{2,a,b}.$$

Now choose $\eta \in D_e(R)$ such that $\eta(x) = 1$ on $[-4, 4]$. Then by Lemma 4.2, we get

$$\int_0^4 |T_\sigma(f)(x)|^2\ d\mu_{a,\,b}(x) \leq \int_0^\infty |T_\sigma(f)(x)|^2\ d\mu_{a,\,b}(x) \leq C\|f\|^2_{2,a,b}.$$

Thus, we have

$$\int_0^\infty |T_\sigma(f)(x)|^2\ d\mu_{a,\,b}(x) \leq \|f\|^2_{2,a,b}.$$

Finally, by using density of $S_e(R)$ in $L_{2,a,b}$, the theorem is proved.

Theorem 4.5. Let $\sigma \in S^0_{a,\,b}$, and T^*_σ be the adjoint operator of T_σ. Then for all $g \in S_e(R)$ and $y\ \geq\ 0, y \notin Supp(g)$, we have

$$T^*_\sigma(g)\ (y) = \int_0^\infty K^*_\sigma(x,\ y) g(x)\ d\mu_{a,\,b}(x),$$

where $K^*_\sigma(y,\ x) = \overline{K_\sigma(x,\ y)}$.

Proof. Let $\gamma(x, y) \in C^{\infty}$ on R^2 with compact support and $\gamma(0, 0) = 1$. Let $\sigma \in S^0_{a,\, b}$, such that

$$\sigma_\epsilon(x,\xi) = \sigma(x,\xi)\,\gamma(\epsilon x, \epsilon\xi),\ 0 < \epsilon < 1.$$

Set $$k_\varepsilon(x,\ z) = \int_0^\infty \sigma_\varepsilon(x,\ \xi)\ j_{\frac{a-b-1}{2}}(z\xi)\ d\mu_{a,\, b}(\xi)\,, \quad x,\ z \geq 0.$$

It is clear that $\sigma_\epsilon(x,\xi) \to \sigma(x,\xi)$ as $\epsilon \to 0$. Now by using the dominated convergence theorem T_σ can be written as

$$T_\sigma(f)\ (x) = \lim_{\varepsilon \to 0} \int_0^\infty \int_0^\infty \sigma_\varepsilon(x,\ \xi) f(y)\ j_{\frac{a-b-1}{2}}(\xi x)\ j_{\frac{a-b-1}{2}}\ (\xi y)\ d\mu_{a,\, b}(\xi)\, d\mu_{a,\, b}(y),$$
$$\forall f \in S_e(R).$$

It is clear that $\sigma_\epsilon \in S^0_{a,\, b}$ and satisfies (2) uniformly in ϵ. Proceeding as in the proof of above lemma, we have for fixed $m > \frac{a-b+1}{2}$,

$$(1+x)^{a-b} |\ B^m_{a,\, b}\ (\sigma_\varepsilon\ (x, \bullet))\ (\xi)|\ \leq \frac{A_m}{(1+\xi)^{2m}}.$$

As $$(-1)^m z^{2m} k_\varepsilon(x,\ z) = \int_0^\infty B^m_{a,\, b}(\sigma_\varepsilon\ (x, \bullet))\ (\xi)\ j_{\frac{a-b-1}{2}}(\xi z)\ d\mu_{a,\, b}(\xi),$$

we have

$$|\ k_\varepsilon(x,\ z)\ \leq \frac{B_m}{z^{2m}}, \quad x \geq\ 0,\ z > 0.$$

Now by dominated convergence theorem, we have

$$k_\epsilon(x,z) \to k_\sigma(x,z) \text{ as } \epsilon \to 0, x \geq 0, z\ >\ 0.$$

Now we can write

$$\int_0^\infty g(x)\ \overline{T_\sigma(f)\ (x)}\ d\mu_{a,\, b}(x)$$
$$= \lim_{\varepsilon \to 0} \int_0^\infty \overline{f(y)} \int_0^\infty \int_0^\infty g(x)\ \overline{\sigma_\varepsilon(x,\ \xi)}\ j_{\frac{a-b-1}{2}}(x\xi)\ j_{\frac{a-b-1}{2}}(y\xi)$$
$$\times\ d\mu_{a,\, b}(x)\ d\mu_{a,\, b}(\xi)\ d\mu_{a,\, b}(y),$$

for all $f, g \in S_e(R)$.

As $$\xi^{2m} \int_0^\infty g(x)\ \sigma_\varepsilon\ (x,\ \xi)\ j_{\frac{a-b-1}{2}}(x\xi)\ d\mu_{a,\, b}(x)$$
$$= \int_0^\infty\ \ j_{\frac{a-b-1}{2}}(x\xi)\ d\mu_{a,\, b}(x),$$

we can obtain for $m > \frac{a-b+1}{2}$,

$$|B^m_{a,\, b}\, g(\sigma_\varepsilon(\bullet,\xi)(x)| \leq \frac{c_m}{(1+x)^{2m}}, x > 0,\ \xi \geq 0.$$

Thus, we can obtain,

$$|\int_0^\infty g(x)\ \sigma_\varepsilon\ (x,\ \xi)\ j_{\frac{a-b-1}{2}}(x\xi)\ d\mu_{a,\, b}(x)\,| \leq \frac{c_m}{(1+x)^{2m}}.$$

where c_m is a constant independent of ϵ. Now by using dominated convergence theorem, Fubini's theorem and the product formula, we can have for $y \geq 0, y \notin Supp(g)$,

$$T_\sigma^*(g)(y) = \lim_{\varepsilon \to 0} \int_0^\infty \int_0^\infty g(x)\, \overline{\sigma_\varepsilon(x,\ \xi)}\ j_{\frac{a-b-1}{2}}(\xi x)\ j_{\frac{a-b-1}{2}}(\xi y)\ d\mu_{a,\,b}(x) d\mu_{a,\,b}(\xi)$$
$$= \lim_{\varepsilon \to 0} \int_0^\infty \int_0^\infty g(x)\, \overline{k_\varepsilon(x,\ z)}\ W_{a,\,b}(x,\ y,\ z)\ d\mu_{a,\,b}(z)\ d\mu_{a,\,b}(x)$$
$$= \int_0^\infty \int_0^\infty g(x)\, \overline{K_\sigma(x,\ z)}\ W_{a,\,b}(x,\ y,\ z)\ d\mu_{a,\,b}(x)\, d\mu_{a,\,b}(z)$$
$$= \int_0^\infty \overline{K_\sigma(x,\ z)}\ \ g(x)\ d\mu_{a,\,b}(x).$$

This proves (23).

Theorem 4.6. Let $\sigma \in S_{a,\,b}^0$, then T_σ can be extended to a bounded operator from $L_{p,a,b}$ into itself for $1 < p < \infty$.

Proof. By density and from Theorem 3.3, the singular integral representation of T_σ can be extended to $L_{p,a,b}$. Now by using Theorem 3.4 and Theorem 4.4, we conclude that T_σ satisfies the conditions (7) and (8) of Theorem 2.1.

Now we can deduce that T_σ is bounded in $L_{p,a,b}$-norm for$1 < p \leq 2$. Further by using (19) and (23), we can apply the Theorem (2.1) to T_σ^*. By duality, T_σ can be extended to a bounded operator from $L_{p,a,b}$ into itself for $1 < p < \infty$. Thus, the theorem is proved.

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