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LAGRANGE SPACE WITH A SPECIAL (γ, β)-METRIC

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In this paper we have obtained the Lagrange Space with a Special (γ, β)-Metric (1.1), where γ is a cubic metric and β is a 1-form metric. We have also calculated the fundamental tensor, its inverse, Euler-Lagrange equations, semispray coefficient, the canonical nonlinear connections and some important properties for this Lagrange space.

Key-Words: Lagrange Spaces with a Special $(\boldsymbol{\gamma}, \boldsymbol{\beta})$-Metric, semispray, Lagrangian, cubic metric, 1-form metric

Introduction

Let $F^n = (M^n, L)$ be a n-dimensional Finsler space where M^n is the n-dimensional differentiable manifold and $L\,(x, y)$ is the Fundamental function, which is a function of point $x = (x^i)$ and element of support $y = (y^i)$. $L\,(x, y)$is positively homogenous of degree one in y.

In the last four decades, various significant generalizations of Finsler space have been obtained. These generalizations have many applications in the field of mechanics, theoretical physics, variational calculus, optimal control, complex analysis, biology, ecology and so forth. The geometry of Lagrange space is one such generalization of the geometry of Finsler spaces which was introduced and studied by Miron [7, 8].

A Lagrange space is a pair $L^n = \big(M, L\,(x, y)\big)$, where M is a smooth manifold and $L\,(x, y)$ is a regular Lagrangian [8]. In the last twenty years this field has attracted worldwide geometers and physicist [2], [19], [9], [14], to work on its development and applications in various disciplines of science.

In the year 2013, Shukla, K. Suresh and Pandey, P. N.,[18] have developed a revised and modified theory of Lagrange Spaces with a Special (γ, β)-Metric, where $\gamma\,(x, y)$ is a cubic metric given by$\gamma^3 = a_{ijk}(x)\, y^i\, y^j y^k$ and $\beta$is a 1-form metric given by $\beta\,(x, y) = b_i(x)\, y^i$.

In this paper, we have studied about Lagrange space in the light of a special (γ,β)- metric whose Lagrangian $\bar{L}(\gamma,\beta)$ is given by

$$\bar{L}(\gamma,\beta) = (\gamma+\beta)^2 \qquad ...(1.1)$$

This Langrangian is too much interesting because the Lagrange space determined by this metric (1.1) is reducible to a Finsler space. For basic notations the terminologies related to Lagrange space is referred to [1], [8].

Preliminaries

Let M be an n-dimensional smooth manifold and let $T\,M$ be its tangent bundle. Let (x^i) and (x^i, y^i) be local coordinates on M and TM respectively. A Lagrangian is a function L: $T\,M \to R$ which is a smooth function on $\widetilde{TM} = T\,M \setminus \{0\}$ and continuous on the null section. The Lagrangian $L\,(x,y)$ is said to be regular if rank $\left(g_{ij}(x,y)\right) = n$,

where $$g_{ij}\,(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j\,L, \quad \dot{\partial}_i = \frac{\partial}{\partial\ y^i} \qquad \dots(2.1)$$

Is a covariant symmetric tensor called the fundamental metric tensor of the Lagrangian $L\,(x,y)$. A Lagrange space is a pair $L^n = \left(M, L\,(x,y)\right), L\,(x,y)$ being a regular Lagrangian whose metric tensor g_{ij} has constant signature on $\widetilde{TM}$.

The integral of action of the Lagrangian $L\,(x,y)$ along a smooth curve $c : [0,1] \to M$ leads to the following Euler-Lagrange equations:

$$E_iL \equiv \frac{\partial\,L}{\partial\,x^i} - \frac{d}{dt}\left(\frac{\partial\,L}{\partial\,y^i}\right) = 0,\ y^i = \frac{d\,x^i}{dt} \qquad ...(2.2)$$

The coefficient of the semispray S of a Lagrange space $L^n = \left(M, L\,(x,y)\right)$ are given by

$$G^i(x,y) = \frac{1}{4}g^{ih}\left(y^k\dot{\partial}_h\partial_k\,L - \partial_h L\right), \partial_k = \frac{\partial}{\partial\ x^k} \qquad ...(2.3)$$

the semispray S is called a canonical semispray as its coefficients depend on $L\,(x,y)$ only.

The coefficients of canonical nonlinear connection $N\left(N^i_j(x,y)\right)$ of a Lagrange space $L^n = \left(M, L\,(x,y)\right)$ are given by

$$N^i_j = \dot{\partial}_j G^i \qquad ...(2.4)$$

$$\gamma^3 = a_{ijk}(x)\,y^i\,y^j y^k \qquad ...(2.5)$$

$$\beta\,(x,y) = b_i(x)\,y^i \qquad ...(2.6)$$

FUNDAMENTAL METRIC TENSOR OF $L^n = (M, L(x, y))$

If we differentiate (2.5) partially with respect to y^j and use the symmetry of a_{rst} in its indices, we obtain

$$\dot{\partial}_j \gamma = \gamma^{-2} a_j(x, y) \qquad \text{... (3.1)}$$

where $\quad a_j(x, y) = a_{jst}(x)\, y^s\, y^t$

Again differentiating (3.1) partially with respect to y^h, using symmetry of $a_{rst}(x)$ in its indices and simplifying, we find

$$\dot{\partial}_j \dot{\partial}_h \gamma = 2\,\gamma^{-2} a_{jh}(x, y) - 2\,\gamma^{-5} a_j a_h\,, \qquad \text{... (3.2)}$$

where $\quad a_{jh} = a_{jhs}\, y^s$

Differentiating (2.6) partially with respect to y^j, we have

$$\dot{\partial}_j \beta = b_j(x) \qquad \text{... (3.3)}$$

Differentiating (3.3) partially with respect to y^h, we get

$$\dot{\partial}_j \dot{\partial}_h \beta = 0 \qquad \text{... (3.4)}$$

Proposition 1. : In a Lagrange space L^n with (γ, β)- metric (1.1) the following holds good:

$$\dot{\partial}_j \gamma = \gamma^{-2} a_j(x, y)$$

$$\dot{\partial}_j \dot{\partial}_h \gamma = 2\,\gamma^{-2} a_{jh}(x, y) - 2\,\gamma^{-5} a_j a_h\,, \qquad \text{... (3.5)}$$

$$\dot{\partial}_j \beta = b_j(x), \quad \dot{\partial}_j \dot{\partial}_h \beta = 0$$

where $\quad a_j(x, y) = a_{jst}(x)\, y^s\, y^t \;,\;\; a_{jh}(x, y) = a_{jhs}\, y^s \qquad$ ...(3.6)

The moments of Lagrangian $L(x, y)$ are given by

$$P_i = \frac{1}{2} \dot{\partial}_i L \qquad \text{... (3.7)}$$

$$L(x, y) = \bar{L}(\gamma, \beta)$$

In our case the Lagrangain $L(x, y)$ is the function of γ and β only. Therefore, we have

$$P_i = \frac{1}{2}\left(\bar{L}_\gamma \dot{\partial}_i \gamma + \bar{L}_\beta \dot{\partial}_i \beta\right) \qquad \text{... (3.8)}$$

where $\bar{L}_\gamma = \dfrac{\partial L}{\partial y} = 2(\gamma + \beta)$, $\bar{L}_\beta = \dfrac{\partial L}{\partial \beta} = 2(\gamma + \beta)$ using equation (3.1), (3.3) in (3.8) we obtain

$$P_i = (\gamma + \beta)\left(\frac{a_i}{\gamma^2} + b_i\right) \qquad \text{... (3.9)}$$

thus, we have the following.

Theorem 2.*In a Lagrange space* (γ, β)- *metric* (1.1) *the moments of Lagrangian* $L\ (x, y)$ *are given by*

$$P_i = \rho\, a_i + \rho_1 b_i\ , \qquad ...(3.10)$$

where

$$\rho = \frac{\gamma + \beta}{\gamma^2}, \qquad ...(3.11)$$

$$\rho_1 = \ (\gamma + \beta), \qquad ...(3.12)$$

the scalars ρ and ρ_1 are called the principal invariants of the space L^n.

Differentiating (3.11) and (3.12), partially with respect to y^j and simplifying we, respectively, have

$$\dot{\partial}_j \rho = -\gamma^{-5}\, a_j(\gamma + 2\beta) + \gamma^{-2} b_j \qquad ...(3.13)$$

$$\dot{\partial}_j \rho_1 = b_j + \gamma^{-2} a_j \qquad ...(3.14)$$

where

$$\bar{L}_{\gamma\gamma} = \frac{\partial^2 \bar{L}}{\partial \gamma^2} = 2, \bar{L}_{\gamma\beta} = \frac{\partial^2 \bar{L}}{\partial \gamma\, \partial \beta} = \frac{\partial^2 \bar{L}}{\partial \beta\, \partial \gamma} = \bar{L}_{\beta\gamma} = 2, \bar{L}_{\beta\beta} = \frac{\partial^2 \bar{L}}{\partial \beta^2} = 2 \quad ...(3.15)$$

Thus, we have the following.

Proposition 3.*The derivative of the principal invariants of a Lagrange space with* (γ, β)- metric (1.1) are given by

$$\dot{\partial}_j \rho = \rho_{-4} a_j + \rho_{-2} b_j\ ,\ \dot{\partial}_j \rho_1 = b_j + \rho_{-2} a_j \qquad ...(3.16)$$

$$\rho_{-4} = -\gamma^{-5}(\gamma + 2\beta)\ , \rho_{-2} = \gamma^{-2} \qquad ...(3.17)$$

the energy of Lagrangian $L\ (x, y)$ is defined as

$$E_L = y^i \dot{\partial}_i\, L - L \qquad ...(3.18)$$

using (1.1) in (3.18), we have

$$E_{\bar{L}} = 2\, y^i (\gamma + \beta)\{\gamma^{-2} a_i + b_i\} - (\gamma + \beta)^2 \qquad ...(3.19)$$

Since γ and β are positively homogenous of degree one in y^i, by virtue of Euler's theorem on homogenous functions, we have

$$\gamma = y^i \dot{\partial}_i\, \gamma\ ,\ \beta = y^i \dot{\partial}_i\, \beta \qquad ...(3.20)$$

in view of (3.20), (3.19) takes the form

$$E_{\bar{L}} = (\gamma + \beta)^2 = \bar{L} \qquad ...(3.21)$$

Thus, we have the following.

Theorem 4. *In a Lagrange space with*(γ, β)- *metric (1.1), the energy of the Lagrangian* $L\ (x, y)$*is given by* $E_{\bar{L}} = \bar{L}$

Now, we find out expression for the fundamental metric tensor $g_{ij}(x, y)$ of a Lagrange space with (γ, β)- metric (1.1). Using (1.1) in (2.1), we have

$$g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L, \quad \dot{\partial}_i = \frac{\partial}{\partial\, y^i} \qquad \text{... (3.22)}$$

In view of proposition 1, (3.22) takes the form

$$g_{ij} = \rho_{-1}a_{ij} + \rho_{-4}a_i\, a_j + \rho_{-2}\left(a_i b_j + a_j b_i\right) + b_i\, b_j \qquad \text{... (3.23)}$$

where $\rho_{-1} = \gamma^{-2}(\gamma + \beta), \quad \gamma^{-2} = \rho_{-2}\rho_{-4} = (-\gamma^{-5} - 2\beta\gamma^{-5})$

Equation (3.23) can be written as

$$g_{ij} = \rho_{-1}a_{ij} + d_i\, d_j \qquad \text{...(3.24)}$$

where $d_i = q_{-2}a_i + b_i$ ... (3.25)

and q_{-2}satisfy

$$\rho_{-2} = q_{-2}\,, \quad \rho_{-4} = (q_{-2})^2 \qquad \text{... (3.26)}$$

Thus, we have the following:

Theorem 5. *The fundamental metric tensor of a Lagrange space with (γ,β)- metric (1.1) is given by.*

$$g_{ij} = \rho_{-1}a_{ij} + d_i\, d_j$$

The following result gives the expression for the inverse of g_{ij} .

Theorem 6. *The inverse g^{ij} of the fundamental metric tensor g_{ij} of a Lagrange space with (γ,β)- metric (1.1) is given by*

$$g^{ij} = \frac{a^{ij}}{\rho_{-1}} - \frac{d_i\, d_j}{\rho_{-1}(\rho_{-1} + d^2)} \qquad \text{... (3.27)}$$

where

(a) $d^i = a^{ir}d_r$ (b) $d^2 = a^{ij}d_i\, d_j$ ...(3.28)

the non-singular matrix g_{ij} given by (3.24).

In [15] Pandey and Chaubey obtained the following

$$\dot{\partial}_i\gamma = \gamma^{-1}y_i \quad \dot{\partial}_i\dot{\partial}_j\gamma = 2\gamma^{-1}a_{ij}(x,y) - \gamma^{-3}y_i y_j \qquad \text{... (3.29)}$$

$$y_i = a_{ij}\,(x,y)y^j$$

$$P_i = \rho y_i + \rho_1 b_i \qquad \text{...(3.30)}$$

where ρ and ρ_1 are, respectively, given by (3.11) and (3.12),

$$\dot{\partial}_i\rho = \rho_{-2}\, y_i + \rho_{-1}b_i \quad ,$$

$$\dot{\partial}_i\rho_1 = \rho_{-i}y_i + b_i \qquad \text{... (3.31)}$$

where $\rho_{-2} = (1/2)\,\gamma^{-2}\left(\bar{L}_{\gamma\gamma} - \gamma^{-1}\bar{L}_\gamma\right) = \gamma^{-2}$, $\rho_{-1} = (1/2)\,\gamma^{-1}\bar{L}_{\gamma\beta} = \gamma^{-2}(\gamma + \beta)$ as given in (3.17)

$$E_{\bar{L}} = \gamma^{-1}\bar{L}_\gamma + \beta\,\bar{L}_\beta - \bar{L}\,, \qquad \text{... (3.32)}$$

$$g_{ij}(x,y) = \rho_{-1}a_{ij} + d_i\, d_j \qquad \text{... (3.33)}$$

where $d_i = q_{-2}\, y_i + b_i$ and $q_{-2} = \rho_{-2}$, satisfy $q_0 q_{-1} = \rho_{-1}, (q_{-2})^2 = \rho_{-4}$,

$$g^{ij} = \left[\frac{a^{ij}}{\rho_{-1}} - \frac{1}{(1+d^2)} d^i d^j\right] \quad \dots (3.34)$$

where $d^i = \frac{1}{\rho} a^{ir} d_r$ and $d^2 = d_i d^i$.

In view of corresponding result obtained by us, these results are erroneous.

EULER-LAGRANGE EQUATIONS

We consider the case of minus polynions and plus polynions separately.

Using (1.1) in (2.2), we obtain

$$E_i(\bar{L}) \equiv \frac{\partial \bar{L}}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial y^i}\right) = 0, y^i = \frac{d\, x^i}{dt} \quad \dots (4.1)$$

For the Langrangian $\bar{L}$ given by equation (1.1), we have

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial y^i}\right) = 2\left(\frac{d\gamma}{dt} + \frac{d\beta}{dt}\right)\frac{\partial \gamma}{\partial y^i} + 2\left(\frac{d\gamma}{dt} + \frac{d\beta}{dt}\right)\frac{\partial \beta}{\partial y^i} + 2(\gamma + \beta)\frac{d}{dt}\left(\frac{\partial \gamma}{\partial y^i}\right)$$
$$+2(\gamma + \beta)\frac{d}{dt}\left(\frac{\partial \beta}{\partial y^i}\right) \quad \dots (4.2)$$

In the view of $\partial_i \bar{L} = \bar{L}_\gamma \partial_i\, \gamma + \bar{L}_\beta \partial_i\, \beta$ and (4.2), (4.1) takes the form

$$E_i(\bar{L}) \equiv 2(\gamma + \beta)E_i(\gamma) + 2(\gamma + \beta)E_i(\beta) - 2\left(\frac{d\gamma}{dt} + \frac{d\beta}{dt}\right)\frac{\partial \gamma}{\partial y^i}$$
$$-2\left(\frac{d\gamma}{dt} + \frac{d\beta}{dt}\right)\frac{\partial \beta}{\partial y^i} \quad \dots (4.3)$$

Since

$$E_i(\gamma^3) = 3\,\gamma^2 E_i(\gamma) - 3\frac{\partial \gamma}{\partial y^i}\frac{d\,\gamma^2}{dt} \quad \dots (4.4)$$

we get

$$E_i(\gamma) = \frac{1}{3}\gamma^{-2} E_i(\gamma^3) + \gamma^{-2}\frac{\partial \gamma}{\partial y^i}\frac{d\,\gamma^2}{dt} \quad \dots (4.5)$$

From

$$E_i(\beta) = 2\, F_{ir} y^r, y^r = \frac{d\, x^r}{dt} \quad \dots (4.6)$$

where

$$F_{ir} = \frac{1}{2}\left(\frac{\partial b_r}{\partial x^i} - \frac{\partial b_i}{\partial x^r}\right) \quad \dots (4.7)$$

Is the electromagnetic tensor field of the potentials b_i. Using (4.5) and (4.6) in (4.3), we obtain

$$E_i(\bar{L}) = \frac{2}{3}(\gamma+\beta)\,\gamma^{-2}E_i(\gamma^3) + 2(\gamma+\beta)\,\gamma^{-2}\frac{\partial\,\gamma}{\partial y^i}\frac{d\,\gamma^2}{dt} + 4(\gamma+\beta)\,F_{ir}y^r$$

$$-2\left(\frac{d\gamma}{dt}+\frac{d\beta}{dt}\right)\frac{\partial\,\gamma}{\partial y^i} - 2\left(\frac{d\gamma}{dt}+\frac{d\beta}{dt}\right)\frac{\partial\,\beta}{\partial y^i} \qquad \dots(4.8)$$

Thus, we have the following.

Theorem 7. *The Euler-Lagrange equations of a Lagrange space with* (γ,β)*- metric (1.1) are given by*

$$E_i(\bar{L}) \equiv \frac{2}{3}\,\rho\,E_i(\gamma^3) + 2\,\rho\,\frac{\partial\,\gamma}{\partial y^i}\frac{d\,\gamma^2}{dt} + 4\,\rho_1 F_{ir}y^r - 2\left(\frac{d\gamma}{dt}+\frac{d\beta}{dt}\right)\frac{\partial\,\gamma}{\partial y^i}$$

$$-2\left(\frac{d\gamma}{dt}+\frac{d\beta}{dt}\right)\frac{\partial\,\beta}{\partial y^i} = 0\,, \qquad \dots(4.9)$$

$$y^i = \frac{d\,x^i}{dt}$$

where $\rho = \gamma^{-2}(\gamma+\beta),\ \rho_{-1} = (\beta+\gamma)$

For the natural parametrization of the curve $c\ :t\ \in [0,1]\ \mapsto x^i(t)\ \in M$ with respect to the cubic metric $\gamma(x,\ dx/dt) = 1$. Thus, we have the following.

Theorem 8. *In the natural parametrization, the Euler-Lagrange equations of a Lagrange space with* (γ,β)*- metric are*

$$E_i(\bar{L}) \equiv \frac{2}{3}\,\rho\,E_i(\gamma^3) + 4\,\rho_1 F_{ir}y^r - 2\frac{d\gamma}{dy^i}\frac{d\beta}{ds} - 2\,\frac{d\beta}{dy^i}\frac{d\beta}{ds} = 0 \qquad \dots(4.10)$$

If β *is constant along the integral curve c of the Euler-Lagrange equations with natural parametrization, then (4.10) takes the form*

$$E_i(\bar{L}) \equiv \frac{2}{3}\,\rho\,E_i(\gamma^3) + 4\,\rho_1 F_{ir}y^r = 0 \qquad \dots(4.11)$$

Thus, we have the following.

Theorem 9. *If* β *is constant along the integral curve of the Euler-Lagrange equations with natural parametrization, then the Euler-Lagrange equations of a Lagrange space with* (γ,β)*- metric (1.1) are given by (4.11).*

CANONICAL SEMISPRAY

In this section, we obtain the coefficients of the canonical semispray of a Lagrange space with (γ,β)- metric (1.1). Using (1.1) in (2.3), we obtain

$$G^i(x,y) = \frac{1}{4}g^{ij}\{y^k\dot{\partial}_h\,\partial_k(\gamma+\beta)^2 - \partial_h\,(\gamma+\beta)^2\},\ \partial_k = \frac{\partial}{\partial x^k} \qquad \dots(5.1)$$

since $\gamma^3 = a_{ijk}(x)\,y^i\,y^j y^k$ and $\beta = b_i(x)\,y^i$, we have

$$\partial_h \gamma = A_h \gamma^{-2}\ ,\quad \partial_h \beta = B_h \qquad ...(5.2)$$

where $A_h = \frac{1}{3}\left(\partial_h a_{ijk}\right) y^i y^j y^k\ ,\ B_h = (\partial_h b_i)\, y^i$ ... (5.3)

using (5.2), (3.11) and (3.12) in $\partial_k \bar{L} = \bar{L}_\gamma \partial_k \gamma + \bar{L}_\beta \partial_k \beta$, we get

$$\partial_k(\gamma + \beta)^2 = 2\,\rho\, A_k + 2\,\rho_1 B_k \qquad ... (5.4)$$

Differentiating (5.4) partially with respect to y^h and simplifying, we have

$$\dot{\partial}_h \dot{\partial}_k (\gamma+\beta)^2 = 2(\rho_{-4} a_h + \rho_{-2} b_h) A_k + 2\,\rho\, A_{kh} + 2\,(b_h + \rho_{-2} a_h) B_k + 2\rho_1 b_{kh} \qquad ... (5.5)$$

where (a) $A_{kh} = \dot{\partial}_h A_k$ (b) $b_{kh} = \dot{\partial}_h B_k$ ...(5.6)

Using (5.4) and (5.5) in (5.1), we obtain

$$G^i = \frac{1}{2} g^{ih} \begin{bmatrix} (\rho_{-4} A_0 + \rho_{-2} B_0) a_h + (\rho_{-2} A_0 + B_0) b_h + \rho\, A_{0h} + \rho_1 b_{0h} \\ -(\rho\, A_h + \rho_1 B_h) \end{bmatrix} \qquad \dots (5.7)$$

where

(i) $A_0 = A_k(x,y) y^k$ (ii) $B_0 = B_k(x,y) y^k$

(iii) $A_{0h} = A_{kh}(x,y) y^k$ (iv) $b_{0h} = b_{kh}(x,y) y^k$ ... (5.8)

Thus, we have the following

Theorem 10. *The local coefficients of canonical semispray of a Lagrange space with* (γ, β)*- metric (1.1) are given by (5.7).*

CANONICAL NONLINEAR CONNECTION

In this section, we obtain the local coefficients of the canonical nonlinear connection of a Lagrange space with (γ, β)- metric (1.1).

Partial differentiation of $g^{ih} g_{is} = \delta^h_s$, with respect to y^j yields.

$$\dot{\partial}_j\, g^{ih} = -2 g^{rh} C^i_{rj} \qquad ... (6.1)$$

If we partially differentiate the quantities appearing in (3.17) and (5.8) with to the respect y^j , we find the following quantities:

$$\dot{\partial}_j \rho_{-4} = \mu_{-7} a_j + \mu_{-5} b_j \qquad \dot{\partial}_j \rho_{-2} = \mu_{-5} a_j$$

$$\dot{\partial}_j A_0 = A_j + A_{0j} \qquad \dot{\partial}_j B_0 = \partial_x b_j y^s + \partial_j b_x y^j$$

$$\dot{\partial}_j A_{0h} = 2\, A_{0hj} + A_{jh} \qquad \dot{\partial}_j b_{0h} = b_{jh} \qquad ... (6.2)$$

where $\mu_{-7} = \gamma^{-8}(6\gamma + 10\beta)\ ,\quad \mu_{-5} = -2\,\gamma^{-5}$ ... (6.3)

and $\mathfrak{S}_{(sj)}$ denotes interchanging indices *s* and *j* and taking sum. Also, we have

$$\dot{\partial}_j a_h = 2\, a_{jh} \qquad ... (6.4)$$

Now, applying (5.7) in (2.4), we get

$$N_j^i = -2\, g^{rh} C_{rj}^i \frac{1}{2}\left[\begin{array}{r} (\rho_{-4}A_0 + \rho_{-2}B_0)a_h + (\rho_{-2}A_0 + B_0)b_h + \rho\, A_{0h} \\ +\rho_1 b_{0h} - (\rho\, A_h + \rho_1 B_h) \end{array}\right]$$

$$\frac{1}{2}\, g^{ih}\left[\begin{array}{r} \{(\dot{\partial}_j\, \rho_{-4})A_0 + \rho_{-4}(\dot{\partial}_j A_0) + (\dot{\partial}_j\, \rho_{-2})B_0 + \rho_{-2}\, \dot{\partial}_j B_0\}\, a_h + (\rho_{-4}A_0 + \rho_{-2}B_0) \\ (\dot{\partial}_j\, a_h) + \{(\dot{\partial}_j\, \rho_{-2})A_0 + \rho_{-2}\, \dot{\partial}_j A_0 + \dot{\partial}_j B_0\}\, b_h + (\dot{\partial}_j \rho)A_{0h} + \rho\, \dot{\partial}_j A_{0h} + \rho_1\, \dot{\partial}_j b_{0h} \\ +b_{0h}\dot{\partial}_j\, \rho_1 - \{(\dot{\partial}_j \rho)A_h + (\rho\, \dot{\partial}_j A_h) + (\dot{\partial}_j\, \rho_1)B_h + \rho_1\, \dot{\partial}_j B_h\} \end{array}\right]$$

...(6.5)

using (3.16), (5.6)– (5.8), (6.1), (6.2) and (6.4) in (6.5) and simplifying, we obtain

$$N_j^i = -2\, C_{rj}^i G^r + \frac{1}{2}\, g^{ih}$$

$$\times\left[\begin{array}{c} \rho_{-4}\{(A_j + A_{0j})\, a_h + (A_{0h} - A_h)\, a_j + 2\, A_0 a_{jh}\} \\ + \rho_{-2}\, \{(A_j + A_{0j})\, b_h + (A_{0h} - A_h)\, b_j + a_j\, b_{0h} + \mathfrak{S}_{(sj)}(\partial_s b_j)y^s a_h + 2B_0\, a_{jh} - a_j\, B_h\} \\ +\mathfrak{S}_{(sj)}(\partial_s b_j)y^s b_h - b_j B_h + b_j b_{0h} + \mathfrak{M}_{(jh)}\{b_{jh}\}\, \rho_1 + \rho(2A_{0hj} + A_{jh} - A_{hj}) \\ +\mu_{-7}A_0 a_j a_h + \mu_{-5}\{A_0 b_j a_h + a_j(B_0\, a_h + A_0\, b_h)\} \end{array}\right]$$

... (6.6)

where $\mathfrak{M}_{(jh)}\{b_{jh}\} = b_{jh} - b_{hj}$.

Thus, we have the following.

Theorem 11. *The local coefficients of canonical nonlinear connection of a Lagrange space with (γ, β)- metric (1.1) are given by (6.6).*

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