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ON CHARACTERISATION OF SEMIGROUP AND ITS INVERSES

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The main purpose here is to show equivalence between quotient group and semigroup properties. The correspondence between the set of associative semigroup and inverse set of matrices has been shown by R. B. Papat. We extend its generalized inverse on semigroups using factorization method. The properties of generalized inverses of a matrix is a structure of a semigroup,which shows that semigroup is not commutative. The modified version of an equivalence relation on isomorphism between the quotient semigroup and the semigroup of projectors have been derived to show that there exist a one-to-one correspondence between the set of matrices and the set of associative semigroups.

Keywords : (Semigroup, Moore-Penrose inverse, Equivalent Matrices, Isomorphism, Semigroup projection)

Introduction

Let K represents the real or the complex field. Let A be an $m \times n$ matrix over K. The generalized inverse inverse of A is an $n \times m$ matrix X over K, which satisfies the matrix equation $AXA = A$. X alsosatisfies the equation $XAX = X$, then X is said to be a reflexive generalized inverse. It is known that there are infinitely sets of $\{1\}$-inverses and $\{1,2\}$ inverses of a matrix A. we establish here that the set of $\{1\}$ inverses of a matrix A a structure of a semigroup and its algebraic properties like factorization and commutativity by using the method the Moore-Penrose inverse. Let us first define an equivalence relation in order to establish an isomorphism between the quotient semigroup and the semigroup of projectors on $R(A)$. We derive the relation between semigroups associated to equivalent matrices and the correspondence between the set $M_{m\times n}(K)$ and the set of the associated semigroups.

Theorem

Let * be a law on $A^{\{1\}}$ defined as follows: for any X, Y $\in A^{\{1\}}$, X * Y = XAY. Then the following statements are equivalent.

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(i) $A^{\{1\}}$ *is a semigroup.*

(ii) *For any X, Y* $\in A^{\{1\}}$ *we have* $X * Y \in A^{\{1,2\}}$

(iii) $A^{\{1,2\}}$ *is an ideal of* $A^{\{1\}}$.

(iv) $P_{AA+} * P_{A+A} = A^{\{1,2\}}$ *and* $P_{AA+} \cap P_{A+A} = \{A^+\}$

(v) *For any X and Y in* $A^{\{1\}}$ *we have*

$$X * Y = Y * X,\ AX \Leftrightarrow AY \text{ and } XA = YA.$$

Proof

(i) A(XAY)A = (AXA) Y A = AY A = A. Then $X * Y = XAY \in A^{\{1\}}$.

The associativity of *is obtained from the associativity of a product of matrices. So $A^{\{1\}}$ is a semigroup.

(ii) Let X and Y be in $A^{\{1\}}$. Since $A(XAY)A = A$ and $(XAY)A(XAY) = X(AYA)(XAY) = X(AXA)Y = XAY$, we obtain $X * Y = XAY \in A^{\{1,2\}}$.

(iii) As $A^{\{1,2\}} \subset A^{\{1\}}$, then 3 is just a simple consequence of 2.

(iv) Since P_{AA+} and P_{A+A} are sub-semigroups of $A^{\{1,2\}}$. Let $A = Q^{-1}\begin{pmatrix} a_r & 0 \\ 0 & 0 \end{pmatrix}P$. For any $Z = P^{-1}\begin{pmatrix} a_r^{-1} & x \\ y & ya_rx \end{pmatrix} Q \in A^{\{1,2\}}$ there exist $Y = P^{-1}\begin{pmatrix} a_r^{-1} & 0 \\ y & 0 \end{pmatrix} Q \in P_{AA+}$ and $X = P^{-1}\begin{pmatrix} a_r^{-1} & x \\ 0 & 0 \end{pmatrix} Q \in P_{A+A}$ such that $Y * X = P^{-1}\begin{pmatrix} a_r^{-1} & 0 \\ y & y0 \end{pmatrix}\begin{pmatrix} a_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_r^{-1} & 0 \\ 0 & 0 \end{pmatrix} Q^{-1} = P^{-1}\begin{pmatrix} a_r^{-1} & x \\ y & ya_rx \end{pmatrix} Q = Z$. Consequently, we obtain $P_{AA+} * P_{A+A} = A^{\{1,2}$. Hence, $P_{A+A} * P_{AA+} = \{A^+\}$. For $Z = p \begin{pmatrix} a_r^{-1} & x \\ y & ya_rx \end{pmatrix} Q^{-1} \in P_{AA+} \cap P_{A+A}$ we obtain $x = 0$ and $y = 0$. Then $Z = P\begin{pmatrix} a_r & 0 \\ 0 & 0 \end{pmatrix}Q^{-1} = A^+$. Therefore $P_{AA+} \cap P_{A+A} = \{A^+\}$. It is replaced by uniqueness of factorization of elements of $A^{\{1,2\}}$.

(v) For any X and Y in $A^{\{1\}}$ we have $XAY = YAX \Rightarrow AXAY = AYAX \Longrightarrow AY = AX$ and $XAY = YAX \Longrightarrow XAYA = YAXA \Longrightarrow XA = YA$. Conversely, $AY = AX \Longrightarrow XAY = XAX$ and $YA = XA \Longrightarrow YAX = XAX$. Hence we obtain $X * Y = Y * X$, $AX \Leftrightarrow AY$ and $XA = YA$.

The commutativity on $A^{\{1\}}$, is not valued but a kind of commutativity modulo projectors is true.

Theorem

Let $A \in M_{m \times n}(K)$, Π be the semigroup of projectors of K^m on $R(A)$. Then the following relations holds.

(i) For any $P \in \Pi$, there exists $X | \in A^{\{1\}}$ such that $AX = P$.

(ii) There is an equivalence relation ~ on $A^{\{1\}}$ such that under the quotient law of *, $A^{\{1\}}/\sim$ is a semigroup isomorphic to Π.

Proof

(i) Let $P \in \Pi$ and $A^{-} \in A^{\{1\}}$. Where AA^{-} is a projector on $R(A)$, we obtain $AA^{-}P = P$ and $PA = A$. Thus, if $X = A^{-}P$, $AXA = AA^{-}PA = AA^{-}A = A$ and $AX = P$ which means that $\Pi = \{AX{=}X \in A^{\{1\}}\}$.

(ii) Let ~ be a relation in $A^{\{1\}}$defined by $X \sim Y$, $AX \Leftrightarrow AY$. Then there exists an equivalence relation in $A^{\{1\}}$. Let X be the canonical map from $A^{\{1\}}$on $A^{\{1\}}/\sim$. Then for every $\mathcal{X}(X)$, $\mathcal{X}(Y) \in A^{\{1\}}/\sim$, the quotient law of ¤ is defined by $\mathcal{X}(X)\, \mathcal{X}(Y) = \mathcal{X}(X * Y)$. we find that $A^{\{1\}}/\sim$ is a semigroup and X is an homomorphism. Which implies that there exists a map X from $A^{\{1\}}/\sim$ to * defined by $\psi(\mathcal{X}(X)) = AX$. We show that ψ is an homomorphism. Since $A(XAY) = (AXA)\, Y = AY$, we obtain $XAY \sim Y$. Then $(XAY) = \mathcal{X}(Y)$. Therefore

$\psi(x(X)\ x(Y)) = \psi(x(X{*}Y)) = \psi(x(XAY)_{=}\ \psi(x(Y)\ AY = (A\ X\ A)Y = (AX)(AY) = \psi(x(X)\psi(x(y))$

Now, since $AX = AY$, it follows that $\mathcal{X}(X) = \mathcal{X}(Y)$ we thus conclude that for any $AX \in \Pi$ there is a unique $\mathcal{X}(X) \in A^{\{1\}}/\sim$ such that $\psi(\mathcal{X}(X)) = AX$ which means that ψ is a bijection.

Theorem

There is a one-to-one correspondence between $\mathrm{M}_{\mathrm{m} \times \mathrm{n}}(\mathrm{K})$ and $\mathrm{M}^{\{1\}}{}_{\mathrm{m} \times \mathrm{n}}(\mathrm{K})$ maps 0 to $M_{n \times m}(K)$ and preserves isomorphisms between semigroups.

Proof

Let ψ be a map from $M_{m \times n}(K)$ onto $M^{\{1\}}{}_{m \times n}(K)$ defined for every $A \in M_{m \times n}(K)$ by $\psi(A) = A^{\{1\}}$. Since $0X0 = 0$ for any $X \in M_{n \times m}(K)$, we get $0^{\{1\}} = M_{n \times m}(K)$. Thus $\psi(0) = M_{n \times m}(K)$. $\Longrightarrow$ we have $rank(A) + rank(B - A) = rank(B)$ and $rank(B) + rank(A - B) = rank(A)$. Thus, we obtain $rank\ (A - B) = 0 = rank\ (B - A)$. Therefore $A = B$. Now, let A and $B \in M_{m \times n}(K)$ such that $B = Q^{-1}AP$. Which implies $B^{\{1\}} = \{ P^{-1}XQ/X \in A^{\{1\}}\} = \varphi(A^{\{1\}})$. Hence, we get $\psi\ (B) = ((\psi\ (A))$.

Theorem

(a) For every matrix $A \in M_{m \times n}\ (K)$ there exists a matrix $A' \in M_{m \times n}\ (K)$ such that $A^{\{1\}} \cap A'^{\{1\}} \neq \phi$.

(b) For any matrices A, $B \in M_{m \times n}\ (K)$ there exists an isomorphism φ from $A^{\{1\}}$ on $\varphi\ (A^{\{1\}})$ such that $\varphi\ (A^{\{1\}} \cap B^{\{1\}} = \phi$.

Proof

(a) Let *rank* $A = r \leq \min(m, n)$. It is sufficient to prove that for $A \in M_{m \times n}(K)$ with *rank* $(A) = r$, there exists a matrix $A' \in M_{m \times n}(K)$ such that $rank(A + A') = rank(A) + rank(A') = \min(m, n)$. wethus find that every $\{1\}$- inverse of $(A + A')$ is a $\{1\}$- inverse of both A and A'. Thus $A^{\{1\}} \cap A' A^{\{1\}} \neq \phi$. Let P and Q be two nonsingular matrices such that $A = Q^{-1}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}P$. Then it is sufficient to take $A' = Q^{-1}\begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix}P$ with $w \in M_{(m-r) \times (n-r)}(K)$ and

$$rank(w) = \min(m, n) - r. \qquad \ldots(1)$$

b) Let us put $rank(A) = r$, $rank(B) = s$. Hence, there exist matrices A' and $B' \in M_{m \times n}(K)$ of ranks $\min(m, n)-r$ and $\min(m, n)-s$ such that $rank(A + A') = rank(A) + rank(A') = \min(m, n)$ and $rank(B + B') = rank(B) + rank(B') = \min(m, n)$. It follows that $(A + A')^{\{1\}} \subset A^{\{1\}} \cap A'^{\{1\}}$, $(B + B')\{1\} \subset A^{\{1\}} \cap B'^{\{1\}}$. Since $A + A'$, $B + B'$ have the same rank, as such they are equivalent. Hence, there exists an isomorphism φ from $A^{\{1\}}$on $\varphi(A^{\{1\}})$ such that $\varphi((A + A')^{\{1\}}) = (B + B')^{\{1\}}$.

$$\Rightarrow \quad \varphi((A+A')^{\{1\}}) \subset \varphi((A)^{\{1\}} \cap A'^{\{1\}}) = \varphi(A^{\{1\}}) \cap \varphi(A'^{\{1\}}) \qquad \ldots(2)$$

and $\quad \varphi((A+A')^{\{1\}}) \subset B^{\{1\}} \cap B'^{\{1\}}$we get $\varphi(A^{\{1\}}) \cap B^{\{1\}} \neq \phi$

Let us take $B = \alpha A$ for a scalar α which is different of 1 and 0, then if $X \in A^{\{1\}} \cap B^{\{1\}}$, we get $\alpha A = \alpha A X \alpha A = \alpha^2 A$. Thus $\alpha \in \{0, 1\}$ which contradicts assumption of the theorem. Consequently we $A^{\{1\}} \cap B^{\{1\}} \neq \phi$.

Theorem

(a) The inclusion is a partial order in $M^{\{a\}}_{m \times n}(K)$ induced by the minus order in the reverse order.

(b) Let $m_0 = \min(m, n)$. For any matrix $A \in M_{m \times n}(K)$ of rank r there exists a sequence of matrices $A = A_r \prec^- A_{r+1} \prec^- \ldots . \prec^- A_{m0}$in $M_{m \times n}(K)$ such that rank (A_r) = rank (A) = r, rank $A_{r+i} = r + i$ for $i = 1, \ldots m_0 - r$. Thus, there exists a sequence $A^{\{1\}}\, m_0 \subset \ldots \subset A^{\{1\}}{}_r = A^{\{1\}}$ and $A^{\{1\}}$ $m_0$is the last term.

Proof

a) For $A, B \in M_{m \times n}(K)$, if $A \prec^- B$, then $rank(B) = rank(A) + rank(B - A)$. It follows that $B^{\{1\}} \subset A^{\{1\}}$. Hence, we obtain the partial order $\subset$ in $M^{\{1\}}_{m \times n}(K)$. b) Let $\{v_1, v_2, \ldots v_r\}$ be a basis of $R(A)$ and $\{v_{r+1}, v_{r+2}, \ldots v_m\}$ be such that the basis of $R(A)$ form a basis for K^m. Let $\{e_1, e_2, \ldots e_n\}$ be a basis for K^n such that $A_r e_j = A e_j = v_j$ for $j = 1, \ldots r$ and $A_r e_j = 0$ for $j = r + 1, \ldots m$ and for $i = 1, \ldots m0 - r$, $A_{r+i} e_j = v_j$ for $j = 1, \ldots r + i$ and $A_{r+i}\, e_j = 0$ for $j = r + i + 1, \ldots m$. In these bases the matrices $A = A_r$ and A_{r+I} are of the form $A = A_r = Q^{-1}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}P$, $A_{r+i} = Q^{-1}\begin{pmatrix} I_{r+i} & 0 \\ 0 & 0 \end{pmatrix}P$ for $i = 1, \ldots m_0 - r$. Then $rank(A_r) = rank(A) = r$, and we obtain the following relation

$$\text{rank}(A_{r+i}) = \text{rank}(A_r)+i = r+i = \text{rank}(A_r) + \text{rank}(A_{r+i}- A_r) \quad \ldots(3)$$

for $i = 1, \ldots m_0- r$. Thus $A = A_r \prec A_{r+1} \prec \ldots\ldots \prec A_{m0}$. By a), we have $A^{\{1\}}{}_{m0} \subset \ldots \subset A^{\{1\}}{}_r = A^{\{1\}}$. $A^{\{1\}}{}_{m0}$ is the last term because A_{m0} is of maximal rank. Hence, the theorem is proved.

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