An International Peer Reviewed Journal of Physical Science
Acta Ciencia Indica. Vol. XLVII-M. No. 1 to 4. 79 (2021)

SAIGO OPERATOR OF FRACTIONAL INTEGRATION OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

VARSHA PARIHAR

(Research Scholar) Department of Mathematics and Statistics
Bhupal Nobles' University, Udaipur (Raj.) Email- varshaaparihar2020@gmail.com

DR. CHANDRAPAL SINGH CHOUHAN

(Supervisor) Department of Mathematics and Statistics
Bhupal Nobles' University, Udaipur (Raj.)

RECEIVED : 16 September, 2021

The aim of this paper is to obtain derivations of some generalized hypergeometric functions for Saigo operator by the application of the generalized fractional integration due to Saigo involving the quadratic transformation formula.

Key-Words: Saigo operator of fractional integration, Gauss hypergeometric function, Pochhammer symbol.

INTRODUCTION AND PRELIMINARIES

The definition of the Gauss's Hypergeometric Series is denoted by $F_1^2\left[\begin{matrix}a,b\\c\end{matrix}|\, z\right]$ which can be further written as

$$F_1^2\left[\begin{matrix}a,b\\c\end{matrix}|\, z\right] = 1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!}+\ \dots..$$

$$F_1^2\left[\begin{matrix}a,b\\c\end{matrix}|\, z\right] = \Sigma_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \qquad \dots (1.1.1)$$

For $a = 1$ and $b = c$ or $b = 1$ and $a = c$, the series (1.1.1) reduced to the well known geometric series and for $a = 0$ and $b = 0$ or both zero, the series becomes unity. If a or b or both are negative integers, the series becomes polynomial.

The natural generalization of the above mentioned functions is the generalized hypergeometric function with p numerator parameters and q denominator parameters denoted by F_q^p and is defined in the following manner.

$$F_q^p\left[\begin{matrix}a_1,a_2\ \dots.\ a_p\\b_1,b_2\ \dots\ b_q\end{matrix},x\right] = \Sigma_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{i=1}^{q}(b_i)_n}\frac{x^n}{n!}$$

Also, if we take $p = q = 1$, the generalized Hypergeometric function reduces to confluent Hypergeometeric function [5], given as

$$F_1^1[a; b|z] = \Sigma_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \qquad \text{... (1.1.2)}$$

where $(a)_n$ is the well known Pochhammer symbol (or the raised or the shifted factorial, since $(1)_n = n!$) defined for $a \in C$ by

$$(a)_n = \begin{cases} a(a+1)\ldots(a+n-1); & n \in N \\ 1; & n = 0 \end{cases} \qquad \text{... (1.1.3)}$$

or in terms of Gamma function

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma a} \quad (a \in C / Z_0^-) \qquad \text{... (1.1.4)}$$

Let $\alpha, \beta, \eta \in C$ and $x \in R^+$. Then the generalized fractional integration due to Saigo [12] is defined as

$$I_{0,x}^{\alpha,\beta,\eta}[f(x)] = \frac{x^{-\alpha-\beta}}{\Gamma\alpha} \int_0^x (x-t)^{\alpha-1} F_1^2\left[\alpha+\beta, -\eta, \alpha; 1-\frac{t}{x}\right] f(t)dt \qquad \text{... (1.1.5)}$$

$$= \frac{d^n}{dx^n} I_{0,x}^{\alpha+\beta-n,\eta-n}[f(x)]$$

where $0 < Re(\alpha)$, $0 < Re(\alpha) + n \leq 1$, $n \in N$ also for σ being real, then function $x^{\sigma-1}$ has the integral formula,

$$I_{0,x}^{\alpha,\beta,\eta}[x^{\sigma-1}] = \frac{\Gamma\sigma\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)} x^{\sigma-\beta-1}$$

where $Re(\sigma) > \max\{0, Re(\beta-\eta)\}$.

Hypergeometric series identities

In this section we obtain derivations of some Hypergeometric functions for Saigo operator.

Theorem 2.1

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}e^{-x}F_1^1\left[\begin{matrix}a\\b\end{matrix}; x\right]\right] = \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)} x^{\sigma-\beta-1}$$

$$F_3^3\left[\begin{matrix}b-a, \sigma, \sigma+\eta-\beta\\ b, \sigma-\beta, \sigma+\eta+\alpha\end{matrix}; -x\right] \qquad \text{... (2.1.1)}$$

Where $Re(\alpha), Re(\beta) > 0, Re(\sigma) = \max\{0, Re(\beta-\eta)\}$; a is a non positive integer.

Proof: We know that

$$e^{-x}F_1^1\left[\begin{matrix}a\\b\end{matrix}; x\right] = F_1^1\left[\begin{matrix}b-a\\b\end{matrix}; -x\right] \qquad \text{... (2.1.2)}$$

and $$I_{0,x}^{\alpha,\beta,\eta}[x^{\sigma-1}] = \frac{\Gamma\sigma\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}$$

Where a is a non positive integer.

Now taking Left side of (2.1.1) and using (2.1.2) we have

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}e^{-x}F_1^1\left[\begin{matrix}a\\b\end{matrix};x\right]\right] = I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}F_1^1\left[\begin{matrix}b-a\\b\end{matrix};-x\right]\right]$$

$$= I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}\Sigma_{n=0}^{\infty}\frac{(b-a)_n}{(b)_n}\frac{(-x)^n}{n!}\right]$$

$$= \left[\Sigma_{n=0}^{\infty}\frac{(b-a)_n(-1)^n}{(b)_n n!}I_{0,x}^{\alpha,\beta,\eta}[x^{\sigma+n-1}]\right]$$

$$= \Sigma_{n=0}^{\infty}\frac{(b-a)_n(-1)^n}{(b)_n n!}\frac{\Gamma(\sigma+n)\Gamma(\sigma+n+\eta-\beta)}{\Gamma(\sigma+n-\beta)\Gamma(\sigma+n+\eta+\alpha)}x^{\sigma+n-\beta-1}$$

$$= \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}$$

$$\Sigma_{n=0}^{\infty}\frac{(b-a)_n}{(b)_n}\frac{(\sigma)_n(\sigma+\eta-\beta)_n}{(\sigma-\beta)_n(\sigma+\eta+\alpha)_n}\frac{(-x)^n}{n!}$$

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}e^{-x}F_1^1\left[\begin{matrix}a\\b\end{matrix};x\right]\right] = \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}F_3^3\left[\begin{matrix}b-a,\sigma,\sigma+\eta-\beta\\b,\sigma-\beta,\sigma+\eta+\alpha\end{matrix};-x\right]$$

This complete the prove of Theorem 2.1.

Theorem 2.2

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}e^{-x}F_2^2\left[\begin{matrix}a,1+d\\b,d\end{matrix};x\right]\right] = \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}F_4^4$$

$$\left[\begin{matrix}b-a-1,\dfrac{d(a-b+1)}{a-d}+1, & \sigma,\sigma+\eta-\beta\\ b,\dfrac{d(a-b+1)}{a-d}+1,\sigma-\beta,\sigma+\eta+\alpha\end{matrix};-x\right] \quad ...(2.2.1)$$

Where $Re(\alpha), Re(\beta) > 0, Re(\sigma) = \max\{0, Re(\beta-\eta)\}$; a is a non positive integer.

Proof: We know that

$$e^{-x}F_2^2\left[\begin{matrix}a,1+d\\b,d\end{matrix};x\right] = F_2^2\left[\begin{matrix}b-a-1, & \dfrac{d(a-b+1)}{a-d}+1\\ b, & \dfrac{d(a-b+1)}{a-d}\end{matrix};-x\right] \quad ...(2.2.2)$$

Now taking left side of (2.2.1) and using (2.2.2) we have

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}e^{-x}F_2^2\left[\begin{matrix}a,1+d\\b,d\end{matrix};x\right]\right] = I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}F_2^2\left[\begin{matrix}b-a-1, \frac{d(a-b+1)}{a-d}+1\\ b, \frac{d(a-b+1)}{a-d}\end{matrix};-x\right]\right]$$

$$= I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}\Sigma_{n=0}^{\infty}\frac{(b-a-1)_n\left(\frac{d(a-b+1)}{a-d}+1\right)_n}{(b)_n\left(\frac{d(a-b+1)}{a-d}\right)_n}\frac{(-x)^n}{n!}\right]$$

$$= \left[\Sigma_{n=0}^{\infty}\frac{(b-a-1)_n\left(\frac{d(a-b+1)}{a-d}+1\right)_n(-1)^n}{(b)_n\left(\frac{d(a-b+1)}{a-d}\right)_n n!}I_{0,x}^{\alpha,\beta,\eta}[x^{\sigma+n-1}]\right]$$

$$= \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}$$

$$\Sigma_{n=0}^{\infty}\frac{(b-a-1)_n\left(\frac{d(a-b+1)}{a-d}+1\right)_n}{(b)_n\left(\frac{d(a-b+1)}{a-d}\right)_n}\frac{(\sigma)_n(\sigma+\eta-\beta)_n}{(\sigma-\beta)_n(\sigma+\eta+\alpha)_n}\frac{(-x)^n}{n!}$$

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}e^{-x}F_2^2\left[\begin{matrix}a,1+d\\ b,d\end{matrix};x\right]\right] = \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}$$

$$F_4^4\left[\begin{matrix}b-a-1,\frac{d(a-b+1)}{a-d}+1, \quad \sigma,\sigma+\eta-\beta\\ b,\frac{d(a-b+1)}{a-d}+1,\sigma-\beta,\sigma+\eta+\alpha\end{matrix};-x\right]$$

This complete the prove of Theorem 2.2.

Theorem 2.3

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}e^{-\frac{x}{2}}F_1^1\left[\begin{matrix}a\\ 2a\end{matrix};x\right]\right] = \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)}x^{\sigma-\beta-1}$$

$$F_5^4\left[\begin{matrix}\frac{\sigma}{2},\frac{\sigma+1}{2},\frac{\sigma+\eta-\beta}{2},\frac{\sigma+\eta-\beta+1}{2}\\ a+\frac{1}{2},\frac{\sigma-\beta}{2},\frac{\sigma-\beta+1}{2},\frac{\sigma+\eta+\alpha}{2},\frac{\sigma+\eta+\alpha+1}{2}\end{matrix};\frac{x^2}{16}\right] \quad ...(2.3.1)$$

where $\mathrm{Re}(\alpha),\mathrm{Re}(\beta) > 0, Re(\sigma) = \max\{0,\mathrm{Re}(\beta-\eta)\}$; a is a non positive integer.

Proof: We know that

$$e^{-\frac{x}{2}}F_1^1\left[\begin{matrix}a\\ 2a\end{matrix};x\right] = F_1^0\left[\begin{matrix}-\\ a+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right] \quad ...(2.3.2)$$

Now taking left side of (2.3.1) and using (2.3.2) we have

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}e^{-\frac{x}{2}}F_1^1\left[\begin{matrix}a\\ 2a\end{matrix};x\right]\right] = I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}F_1^0\left[\begin{matrix}-\\ a+\frac{1}{2}\end{matrix};\frac{x^2}{16}\right]\right]$$

$$= I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}\Sigma_{n=0}^{\infty}\frac{1}{\left(a+\frac{1}{2}\right)_n}\frac{(x)^{2n}}{n!\,(16)^n}\right]$$

$$= \left[\Sigma_{n=0}^{\infty} \frac{1}{\left(a+\frac{1}{2}\right)_n n!\,(16)^n} I_{0,x}^{\alpha,\beta,\eta}[x^{\sigma+2n-1}]\right]$$

$$= \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)} x^{\sigma-\beta-1}$$

$$\Sigma_{n=0}^{\infty} \frac{1}{\left(a+\frac{1}{2}\right)_n} \frac{\left(\frac{\sigma}{2}\right)_n \left(\frac{\sigma+1}{2}\right)_n \left(\frac{\sigma+\eta-\beta}{2}\right)_n \left(\frac{\sigma+\eta-\beta+1}{2}\right)_n}{\left(\frac{\sigma-\beta}{2}\right)_n \left(\frac{\sigma-\beta+1}{2}\right)_n \left(\frac{\sigma+\eta+\alpha}{2}\right)_n \left(\frac{\sigma+\eta+\alpha+1}{2}\right)_n} \frac{(x)^{2n}}{n!\,(16)^n}$$

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1} e^{-\frac{x}{2}} F_1^1\left[\begin{matrix} a \\ 2a \end{matrix}; x\right]\right] = \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)} x^{\sigma-\beta-1}$$

$$F_5^4\left[\begin{matrix} \frac{\sigma}{2}, \frac{\sigma+1}{2}, \frac{\sigma+\eta-\beta}{2}, \frac{\sigma+\eta-\beta+1}{2} \\ a+\frac{1}{2}, \frac{\sigma-\beta}{2}, \frac{\sigma-\beta+1}{2}, \frac{\sigma+\eta+\alpha}{2}, \frac{\sigma+\eta+\alpha+1}{2} \end{matrix}; \frac{x^2}{16}\right]$$

This complete the prove of Theorem 2.3.

Theorem 2.4

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1} e^{-x} \left\{F_1^1\left[\begin{matrix} a \\ 2a \end{matrix}; x\right]\right\}^2\right] = \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)} x^{\sigma-\beta-1}$$

$$F_6^5\left[\begin{matrix} a, \frac{\sigma}{2}, \frac{\sigma+1}{2}, \frac{\sigma+\eta-\beta}{2}, \frac{\sigma+\eta-\beta+1}{2} \\ a+\frac{1}{2}, 2a, \frac{\sigma-\beta}{2}, \frac{\sigma-\beta+1}{2}, \frac{\sigma+\eta+\alpha}{2}, \frac{\sigma+\eta+\alpha+1}{2} \end{matrix}; \frac{x^2}{4}\right] \quad \text{... (2.4.1)}$$

where $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0, Re(\sigma) = \max\{0, \mathrm{Re}(\beta-\eta)\}$; a is a non positive integer.

Proof: We know that

$$e^{-x}\left\{F_1^1\left[\begin{matrix} a \\ 2a \end{matrix}; x\right]\right\}^2 = F_2^1\left[\begin{matrix} a \\ a+\frac{1}{2}, 2a \end{matrix}; \frac{x^2}{4}\right] \quad \text{... (2.4.2)}$$

Now taking left side of (2.4.1) and using (2.4.2) we have

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1} e^{-x} \left\{F_1^1\left[\begin{matrix} a \\ 2a \end{matrix}; x\right]\right\}^2\right] = I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1} F_2^1\left[\begin{matrix} a \\ a+\frac{1}{2}, 2a \end{matrix}; \frac{x^2}{4}\right]\right]$$

$$= I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1} \Sigma_{n=0}^{\infty} \frac{(a)_n}{\left(a+\frac{1}{2}\right)_n (2a)_n} \frac{(x)^{2n}}{n!\,(4)^n}\right]$$

$$= \left[\Sigma_{n=0}^{\infty} \frac{(a)_n}{\left(a+\frac{1}{2}\right)_n (2a)_n n!\,(4)^n} I_{0,x}^{\alpha,\beta,\eta}[x^{\sigma+2n-1}]\right]$$

$$= \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)} x^{\sigma-\beta-1}$$

$$\Sigma_{n=0}^{\infty} \frac{1}{\left(a+\frac{1}{2}\right)_n} \frac{(a)_n \left(\frac{\sigma}{2}\right)_n \left(\frac{\sigma+1}{2}\right)_n \left(\frac{\sigma+\eta-\beta}{2}\right)_n \left(\frac{\sigma+\eta-\beta+1}{2}\right)_n}{(2a)_n \left(\frac{\sigma-\beta}{2}\right)_n \left(\frac{\sigma-\beta+1}{2}\right)_n \left(\frac{\sigma+\eta+\alpha}{2}\right)_n \left(\frac{\sigma+\eta+\alpha+1}{2}\right)_n} \frac{(x)^{2n}}{n!\,(4)^n}$$

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1} e^{-x} \left\{F_1^1\left[\begin{matrix} a \\ 2a \end{matrix}; x\right]\right\}^2\right] = \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)} x^{\sigma-\beta-1}$$

$$F_6^5\left[\begin{matrix} a, \frac{\sigma}{2}, \frac{\sigma+1}{2}, \frac{\sigma+\eta-\beta}{2}, \frac{\sigma+\eta-\beta+1}{2} \\ a+\frac{1}{2}, 2a, \frac{\sigma-\beta}{2}, \frac{\sigma-\beta+1}{2}, \frac{\sigma+\eta+\alpha}{2}, \frac{\sigma+\eta+\alpha+1}{2} \end{matrix}; \frac{x^2}{4}\right]$$

This complete the prove of Theorem 2.4.

Theorem 2.5

$$I_{0,x}^{\alpha,\beta,\eta}[x^{\sigma-1}\{(1-x)^{-a}+(1+x)^{-a}\}] = 2\frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)} x^{\sigma-\beta-1}$$

$$F_3^4\left[\begin{matrix} \frac{a}{2}, \frac{a}{2}+\frac{1}{2}, \sigma, \sigma+\eta-\beta \\ \frac{1}{2}, \sigma-\beta, \sigma+\eta+\alpha \end{matrix}; x\right] \quad \ldots (2.5.1)$$

where $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0, Re(\sigma) = \max\{0, \mathrm{Re}(\beta-\eta)\}$; a is a non positive integer.

Proof: We know that

$$(1-x)^{-a}+(1+x)^{-a} = 2F_1^2\left[\begin{matrix} \frac{a}{2}, \frac{a}{2}+\frac{1}{2} \\ \frac{1}{2} \end{matrix}; x\right] \quad \ldots (2.5.2)$$

Now taking left side of (2.5.1) and using (2.5.2) we have

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}.\,2F_1^2\left[\begin{matrix} \frac{a}{2}, \frac{a}{2}+\frac{1}{2} \\ \frac{1}{2} \end{matrix}; x\right]\right] = 2\, I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1} F_1^2\left[\begin{matrix} \frac{a}{2}, \frac{a}{2}+\frac{1}{2} \\ \frac{1}{2} \end{matrix}; x\right]\right]$$

$$= 2 I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1} \Sigma_{n=0}^{\infty} \frac{\left(\frac{a}{2}\right)_n \left(\frac{a}{2}+\frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_n} \frac{(x)^n}{n!}\right]$$

$$= 2\left[\Sigma_{n=0}^{\infty} \frac{\left(\frac{a}{2}\right)_n \left(\frac{a}{2}+\frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_n n!} I_{0,x}^{\alpha,\beta,\eta}[x^{\sigma+n-1}]\right]$$

$$= 2\frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)} x^{\sigma-\beta-1} \Sigma_{n=0}^{\infty} \frac{\left(\frac{a}{2}\right)_n \left(\frac{a}{2}+\frac{1}{2}\right)_n (\sigma)_n(\sigma+\eta-\beta)_n}{\left(\frac{1}{2}\right)_n (\sigma-\beta)_n(\sigma+\eta+\alpha)_n} \frac{x^n}{n!}$$

$$I_{0,x}^{\alpha,\beta,\eta}\left[x^{\sigma-1}e^{-x}\left\{F_1^1\left[\begin{matrix} a \\ 2a \end{matrix}; x\right]\right\}^2\right]$$

$$= 2\frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\eta+\alpha)} x^{\sigma-\beta-1} F_3^4\left[\begin{matrix} \frac{a}{2}, \frac{a}{2}+\frac{1}{2}, \sigma, \sigma+\eta-\beta \\ \frac{1}{2}, \sigma-\beta, \sigma+\eta+\alpha \end{matrix}; x\right]$$

This complete the prove of Theorem 2.5.

ACKNOWLEDGMENTS:

The authors thanks to the anonymous referees for valuable comments and useful suggestions, improving the quality of the article.

REFERENCES

1. G. E. Andrews and C. Larry, Special Funtion for Engineers and Applied Mathematicians, Macmillan, New York (1985).
2. G. E. Andrews and D. Stanton, Determinants in plane partition enumeration, *Europ. J. Combin*., 19, 273-282 (1998).
3. W. N. Bailey, Generalized hypergeometric series, Cambridge University Press, Cambridge (1935).
4. W.N. Bailey, Products of generalized hypergeometric series, *Proc. London Math. Soc.*, 28, No. 2, 242-254 (1928).
5. Bruce C. Berndt, Ramanujan’s Notebooks, Part III, Springer-Verlag (1989).
6. Dinesh Narayan Vyas, Ph.D Thesis, Jai Narayan Vyas University, Jodhpur (India) (1993).
7. D. Earl Rainville, *Special Function*, Macmillan, New York (1960).
8. G. Gasper and M. Rahman, Basic hypergeometric series, *Encyclopedia of Mathematics and Its Applications*, 35, Gambridge University Press, Gam- bridge (1990).
9. Lal Sahab Singh and Dharmendra Kumar Singh, Saigo operator of fractional integration for product of two hypergeometric functions, *Acta Ciencia Indica*, Vol. XXXII M, No. 3 (2006), 1067-1072.
10. Y. L. Luke, *The Special Functions and Their Approximations*, Vol. 1, Academic Press, London (1969).
11. M. Rahman and A. Verma, Quadratic transformation formulas for basic hypergeometric series, Trans. Amer. Math. Soc., 335, 277-302 (1993).
12. M. Saigo, A remark on integral operators involving the Gauss hypergeometric function, Rep. College General Ed., Kyushu Univ., 11, 135-143 (1978).
13. V. N. Singh, The basic analogues of identities of the Cayley-Orr type, J.London Math. Soc., 34, 15-22 (1959).
14. L. J. Slater, Generalized hypergeometric functions, Cambridge University Press, Cambridge (1966).

15. K. Srinivasa Rao, H. D. Doebner and P. Nattermann, Group Theoretical basis for some transformations of generalized hypergeometric series and the symmetries of the 3-j and 6-j coefficients, Proc. of the 5th Wigner Symposium, Ed. by P. Kasperkovitz and D. Grau, World Scientific 97-99 (1998).

❑