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GENERALIZATION OF MITTAG-LEFFLER FUNCTION AND ITS PROPERTIES

NANCY SOLANKI

(Research Scholar) Department of Mathematics and Statistics
Bhupal Nobles' University, Udaipur (Raj.)
Email- nancysolanki1818@gmail.com

DR. CHANDRAPAL SINGH CHOUHAN

(Supervisor)
Department of Mathematics and Statistics, Bhupal Nobles' University, Udaipur (Raj.)

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In this paper our aim is to introduce and investigate the function

$$E_{\alpha,\beta}^{\gamma,\delta,q,r}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(\delta)_{rn} z^n}{\Gamma(\alpha n+\beta)\, n!\, n!}$$

where for $\boldsymbol{\alpha, \beta, \gamma, \delta} \in \mathbb{C}$; $\mathbf{Re}(\boldsymbol{\alpha}) > \mathbf{0}, \boldsymbol{Re}(\boldsymbol{\beta}) > \mathbf{0}, \boldsymbol{Re}(\boldsymbol{\gamma}) > \mathbf{0}$, $\boldsymbol{Re}(\boldsymbol{\delta}) > 0$ and $\mathbf{q}, \mathbf{r} \in (\mathbf{0}, \mathbf{1}) \cup \mathbf{N}$. This is a generalization of the exponential function $\mathbf{exp}(\mathbf{z})$, the confluent hypergeometric function $\boldsymbol{\Phi}(\boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\alpha}; \mathbf{z})$, the Mittag-Leffler function $\mathbf{E}_{\boldsymbol{\alpha}}(\mathbf{z})$, the Wiman's function $\mathbf{E}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(\mathbf{z})$ and the function $\mathbf{E}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{\boldsymbol{\gamma}}(\mathbf{z})$ defined by Prabhakar and the function $\mathbf{E}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{\boldsymbol{\gamma},\mathbf{q}}(\mathbf{z})$ Shukla and Prajapati with its various properties. For the function $\mathbf{E}_{\boldsymbol{\alpha},\boldsymbol{\beta}}^{\boldsymbol{\gamma},\boldsymbol{\delta},\mathbf{q},\mathbf{r}}(\mathbf{z})$ including usual differentiation and integration, generalised hypergeometric series form, Mellin–Barnes integral representation with their several special cases are obtained and its relationship with Laguerre polynomials, Fox H-function and Wright hypergeometric function is also established.

Keywords: Confluent hypergeometric function; Euler transform; Fox H-function; Mellin transform; Mittag-Leffler function; Whittaker transform; Wiman's function; Wright hypergeometric function.

INTRODUCTION

In 1903, the Swedish mathematician Gosta Mittag-Leffler [6] introduced the function $E_\alpha(z)$, defined as follows,

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} \qquad \ldots (1.1)$$

where z is a complex variable and $\Gamma(s)$ is a Gamma function, $\alpha \geq 0$. The Mittag-Leffler function is a direct generalisation of the exponential function to which it reduces for $\alpha = 1$. For $0 < \alpha < 1$ it interpolates between the pure exponential and a hypergeometric function $\frac{1}{1-z}$. Its importance is realized during the last two decades due to its involvement in the problems of physics, chemistry, biology, engineering and applied sciences. Mittag-Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equations.

The generalization of $E_\alpha(z)$ was studied by Wiman [12] in 1905 and he defined the function as

$$E_{\alpha,\beta}(z) = \Sigma_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad \ldots (1.2)$$

where $\alpha, \beta \in C$; $Re(\alpha) > 0, Re(\beta) > 0$ and which is known as Wiman's function or for $\beta = 1$, generalised Mittag-Leffler function as $E_{\alpha,1}(z) = E_\alpha(z)$.

In 1971, Prabhakar [7] introduced the function $E_{\alpha,\beta}^{\gamma}(z)$ in the form of

$$E_{\alpha,\beta}^{\gamma}(z) = \Sigma_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta)\, n!} \quad \ldots (1.3)$$

where $\alpha, \beta, \gamma \in C$; $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $(\gamma)_n$ is the Pochhammer symbol (Rainville [8])

$$(\gamma)_0 = 1, (\gamma)_n = \gamma(\gamma + 1)(\gamma + 2) \cdots (\gamma + n - 1).$$

The function $E_{\alpha,\beta}^{\gamma}(z)$ is a most natural generalisation of the exponential function $\exp(z)$, Mittag-Leffler function $E_\alpha(z)$ and Wiman's function $E_{\alpha,\beta}(z)$. Gorenflo et al. [2 ,3], Kilbas and Saigo [4,9] investigated several properties and applications of (1.1)–(1.3).

In 2007, Shukla and Prajapati [11] introduced the function $E_{\alpha,\beta}^{\gamma,q}(z)$ in the form of

$$E_{\alpha,\beta}^{\gamma,q}(z) = \Sigma_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n + \beta)\, n!} \quad \ldots (1.4)$$

which is defined for $\alpha, \beta, \gamma \in C$; $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$ and $q \in (0,1) \cup N$.

In continuation of this study, we investigate the function $E_{\alpha,\beta}^{\gamma,\delta,q,r}(z)$ in the form of

$$E_{\alpha,\beta}^{\gamma,\delta,q,r}(z) = \Sigma_{n=0}^{\infty} \frac{(\gamma)_{qn} (\delta)_{rn} z^n}{\Gamma(\alpha n + \beta)\, n!\, n!} \quad \ldots (1.5)$$

which is defined for $\alpha, \beta, \gamma, \delta \in C$; $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\delta) > 0$ and $q, r \in (0,1) \cup N$ also where $(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ and $(\delta)_{rn} = \frac{\Gamma(\delta+rn)}{\Gamma(\delta)}$ denotes the generalized Pochhammer symbol which in particular reduces to $q^{qn} \prod_{s=1}^{q} \left(\frac{(\gamma+s-1)}{q}\right)_n$ if $q \in N$ and

$r^{rn} \prod_{s=1}^{r} \left(\frac{(\delta+s-1)}{r}\right)_n$. The function $E_{\alpha,\beta}^{\gamma,\delta,q,r}(z)$ converges absolutely for all z if $q, r < Re(\alpha) + 1$ and for $|z| < 1$ if $q, r = Re(\alpha) + 1$. It is an entire function of order $(\mathrm{Re}\,\alpha)^{-1}$.

(1.5) is a generalization of all above functions defined by Eqs. (1.1)–(1.4).

Remarks:

- If we take $\beta = 1, \gamma = 1, \delta = 1, q = 1, r = 1$ in (1.5) then we get (1.1)
- If we take $\gamma = 1, \delta = 1, q = 1, r = 1$ in (1.5) then we get (1.2)
- If we take $q = 1,\ \delta = 1,\ r = 1$ in (1.5) then we get (1.3)
- If we take $\delta = 1,\ r = 1$ in (1.5) then we get (1.4)
- is special case of (1.2) for $\beta = 1$.
- is special case of (1.3) for $\gamma = 1$.
- is special case of (1.3) for $\beta = \gamma = 1$.
- is special case of (1.4) for $q = 1$.
- is special case of (1.4) for $q = \gamma = 1$.
- is special case of (1.4) for $\beta = q = \gamma = 1$.
- is special case of (1.5) for $\delta = r = 1$.
- is special case of (1.5) for $q = \delta = r = 1$.
- is special case of (1.5) for $\gamma = q = \delta = r = 1$.
- is special case of (1.5) for $\beta = \gamma = q = \delta = r = 1$.

$$E_{\alpha,1}^{\gamma,1,q,1}(z) = E_{\alpha,1}^{\gamma,1}(z) = E_{\alpha,1}^{1}(z) = E_{\alpha,1}(z) = E_{\alpha}(z)$$

Also

$$(1.5) \Rightarrow (1.4) \Rightarrow (1.3) \Rightarrow (1.2) \Rightarrow (1.1)$$

But in general,

$$(1.5) \nLeftarrow (1.4) \nLeftarrow (1.3) \nLeftarrow (1.2) \nLeftarrow (1.1)$$

This is also known as the Pochhammer–Barnes confluent hypergeometric function and is defined as

$$\Phi(a, b;\ z) = F_1^1(a, b;\ z) = \Sigma_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}, \qquad \dots (1.6)$$

where $b \neq 0$ or a negative integer is convergent for all finite z.

- Generalised Laguerre polynomials (Rainville [8]):

These are also known as Sonine polynomials and are defined as

$$L(\alpha)n\ (x) = \frac{(1\ +\alpha)_n}{n!}\ F_1^1\left[\begin{matrix}-n\\1+\alpha\end{matrix};\ x\right], \qquad \dots (1.7)$$

in which *n* is a non-negative integer.

- Wright generalised hypergeometric function (Srivastava and Manocha [11]):

This is denoted by Ψ_q^p and is defined as

$$\Psi_q^p\left[\begin{matrix}(\alpha_1, A_1), \dots, (\alpha_p, A_p)\\ z\ (\beta 1, B1), \dots, (\beta q, Bq\ )\end{matrix}; z\right] = \Sigma_{n=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(\alpha_i + A_i n)}{\prod_{i=1}^{q}\Gamma(\beta_i + B_i n)}\frac{z^n}{n!} \qquad \dots (1.8)$$

$$= H_{p,q+1}^{1,p}\left[-z\left|\begin{matrix}(1-\alpha_1, A_1), \dots, (1-\alpha_p, A_p)\\(0,1), (1-\beta_1, B_1), \dots, (1-\beta_q, B_q)\end{matrix}\right.\right], \qquad \dots (1.9)$$

where $$H_{p,q}^{m,n}\left[-z\left|\begin{matrix}(\alpha_1, A_1), \dots, (\alpha_p, A_p)\\(\beta_1, B_1), \dots, (\beta_q, B_q)\end{matrix}\right.\right]$$

denotes the Fox H −function.

Carlitz [1] used the following formula:

$$(a\ +b)_m\ = \Sigma_{r=0}^{m}\ (a)_r\ (b)_{m-r}. \qquad \dots (1.10)$$

BASIC PROPERTIES OF THE FUNCTION $E_{\alpha,\beta}^{\gamma,\delta,q,r}$

As a consequence of the definitions (1.1) to (1.5) the following results hold:

Theorem 2.1. If $\alpha, \beta, \gamma, \delta\ \in\ C$, $Re(\alpha) > 0;\ q, r\ \in\ N$ then

$$E_{\alpha,\beta}^{\gamma,\delta,q,r}(z)\ = \beta E_{\alpha,\beta+1}^{\gamma,\delta,q,r}\ (z) + \alpha z\frac{d}{dz}E_{\alpha,\beta+1}^{\gamma,\delta,q,r}\ (z)\ , \qquad \dots (2.1.1)$$

$$E_{\alpha,\beta-\alpha}^{\gamma,\delta,q,r}\ (z) - E_{\alpha,\beta-\alpha}^{\gamma-1,\delta-1,q,r}\ (z)\ = zr\Sigma_{n=0}^{\infty}\ \frac{(\gamma-1)_{qn+q-1}(\delta)_{rn+r-}\ z^n}{\Gamma(\alpha n+\beta)\ (n+1)!\ n!}$$

$$+zq\Sigma_{n=0}^{\infty}\ \frac{(\gamma)_{qn+q-1}(\delta-1)_{rn+r-}\ z^n}{\Gamma(\alpha n+\beta)\ n!\ (n+1)!}$$

$$+zqr\Sigma_{n=0}^{\infty}\ \frac{(\gamma)_{qn+q-1}(\delta)_{rn+r-1}z^n}{\Gamma(\alpha n+\beta)\ (n+1)!\,(n+1)!}, \qquad \dots (2.1.2)$$

in particular,

$$E_{\alpha,\beta-\alpha}^{\gamma,\delta}\ (z) - E_{\alpha,\beta-\alpha}^{\gamma-1,\delta-1}\ (z) = z\Sigma_{n=0}^{\infty}\ \frac{(\gamma-1)_n(\delta)_n z^n}{\Gamma(\alpha n+\beta)\ (n+1)!\ n!} + z\Sigma_{n=0}^{\infty}\ \frac{(\gamma)_n(\delta-1)_n z^n}{\Gamma(\alpha n+\beta)\ n!\ (n+1)!}$$

$$+z\Sigma_{n=0}^{\infty}\ \frac{(\gamma)_n(\delta)_n z^n}{\Gamma(\alpha n+\beta)\ (n+1)!\,(n+1)!}\ . \qquad \dots (2.1.3)$$

Proof: We know that

$$E_{\alpha,\beta}^{\gamma,\delta,q,r}(z) = \Sigma_{n=0}^{\infty} \frac{(\gamma)_{qn}(\delta)_{rn} z^n}{\Gamma(\alpha n+\beta)\, n!\; n!}$$

Let us consider

$$\alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{\gamma,\delta,q,r}(z) = \alpha z \frac{d}{dz} \Sigma_{n=0}^{\infty} \frac{(\gamma)_{qn}(\delta)_{rn} z^n}{\Gamma(\alpha n+\beta+1)\, n!\; n!}$$

$$\alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{\gamma,\delta,q,r}(z) = \Sigma_{n=0}^{\infty} \frac{(\alpha n+\beta-\beta)(\gamma)_{qn}(\delta)_{rn} z^n}{\Gamma(\alpha n+\beta+1)\, n!\; n!}$$

$$\alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{\gamma,\delta,q,r}(z) = \Sigma_{n=0}^{\infty} \frac{(\gamma)_{qn}(\delta)_{rn} z^n}{\Gamma(\alpha n+\beta)\, n!\; n!} - \beta\, \Sigma_{n=0}^{\infty} \frac{(\gamma)_{qn}(\delta)_{rn} z^n}{\Gamma(\alpha n+\beta+1)\, n!\; n!}$$

$$\alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{\gamma,\delta,q,r}(z) = E_{\alpha,\beta}^{\gamma,\delta,q,r}(z) - \beta E_{\alpha,\beta+1}^{\gamma,\delta,q,r}(z)$$

$$E_{\alpha,\beta}^{\gamma,\delta,q,r}(z) = \beta E_{\alpha,\beta+1}^{\gamma,\delta,q,r}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{\gamma,\delta,q,r}(z)$$

This proves (2.1.1).

Again consider L.H.S of (2.1.2) we have

$$E_{\alpha,\beta-\alpha}^{\gamma,\delta,q,r}(z) - E_{\alpha,\beta-\alpha}^{\gamma-1,\delta-1,q,r}(z) = \Sigma_{n=0}^{\infty} \frac{(\gamma)_{qn}(\delta)_{rn} z^n}{\Gamma(\alpha n+\beta-\alpha)\, n!\; n!} - \Sigma_{n=0}^{\infty} \frac{(\gamma-1)_{qn}(\delta-1)_{rn} z^n}{\Gamma(\alpha n+\beta-\alpha)\, n!\; n!}$$

$$= \Sigma_{n=0}^{\infty} \frac{(\gamma)_{qn}(\delta)_{rn} z^n}{\Gamma(\alpha(n-1)+\beta)\, n!\; n!} - \Sigma_{n=0}^{\infty} \frac{(\gamma-1)_{qn}(\delta-1)_{rn} z^n}{\Gamma(\alpha(n-1)+\beta)\, n!\; n!}$$

$$= \Sigma_{n=0}^{\infty} \frac{\left[(\gamma)_{qn}(\delta)_{rn} - (\gamma-1)_{qn}(\delta-1)_{rn}\right] z^n}{\Gamma(\alpha(n-1)+\beta)\, n!\; n!}$$

$$= \Sigma_{n=0}^{\infty} \frac{[\gamma rn + qn\delta + qnrn - qn - rn]\left\{(\gamma)_{qn-1}(\delta)_{rn-1}\right\} z^n}{\Gamma(\alpha(n-1)+\beta)\, n!\; n!}$$

$$= \Sigma_{n=0}^{\infty} \frac{[(\gamma-1)rn + qn(\delta-1) + qnrn]\left\{(\gamma)_{qn-1}(\delta)_{rn-1}\right\} z^n}{\Gamma(\alpha(n-1)+\beta)\, n!\; n!}$$

$$= \Sigma_{n=0}^{\infty} \frac{(\gamma-1)rn(\gamma)_{qn-1}(\delta)_{rn-1} z^n}{\Gamma(\alpha(n-1)+\beta)\, n!\; n!} + \Sigma_{n=0}^{\infty} \frac{qn(\delta-1)(\gamma)_{qn-1}(\delta)_{rn-1} z^n}{\Gamma(\alpha(n-1)+\beta)\, n!\; n!}$$

$$+\Sigma_{n=0}^{\infty} \frac{qnrn(\gamma)_{qn-1}(\delta)_{rn-1} z^n}{\Gamma(\alpha(n-1)+\beta)\, n!\; n!}$$

$$= r\Sigma_{n=1}^{\infty} \frac{(\gamma-1)_{qn-1}(\delta)_{rn-1} z^n}{\Gamma(\alpha(n-1)+\beta)\, (n-1)!\; n!} + q\Sigma_{n=1}^{\infty} \frac{(\gamma)_{qn-1}(\delta-1)_{rn-1} z^n}{\Gamma(\alpha(n-1)+\beta)\, n!\; (n-1)!}$$

$$+qr\Sigma_{n=1}^{\infty} \frac{(\gamma)_{qn-1}(\delta)_{rn-1} z^n}{\Gamma(\alpha(n-1)+\beta)\, (n-1)!\, (n-1)!}$$

Put $(n+1)$ in place of n

$$= zr\Sigma_{n=0}^{\infty} \frac{(\gamma-1)_{qn+q-}\; (\delta)_{rn+r-1} z^n}{\Gamma(\alpha n+\beta)\, (n+1)!\; n!}$$

$$+zq\Sigma_{n=0}^{\infty}\frac{(\gamma)_{qn+q-1}(\delta-1)_{rn+r-1}z^n}{\Gamma(\alpha n+\beta)\,n!\,(n+1)!}+zqr\Sigma_{n=0}^{\infty}\frac{(\gamma)_{qn+q-}\,(\delta)_{rn+r-}\,z^n}{\Gamma(\alpha n+\beta)\,(n+1)!\,(n+1)!}$$

$$E_{\alpha,\beta-\alpha}^{\gamma,\delta,q,r}(z)-E_{\alpha,\beta-\alpha}^{\gamma-1,\delta-1,q,r}(z)=zr\Sigma_{n=0}^{\infty}\frac{(\gamma-1)_{qn+q-1}(\delta)_{rn+r-1}z^n}{\Gamma(\alpha n+\beta)\,(n+1)!\,n!}$$

$$+zq\Sigma_{n=0}^{\infty}\frac{(\gamma)_{qn+q-1}(\delta-1)_{rn+r-}\,z^n}{\Gamma(\alpha n+\beta)\,n!\,(n+1)!}+zqr\Sigma_{n=0}^{\infty}\frac{(\gamma)_{qn+q-1}(\delta)_{rn+r-}\,z^n}{\Gamma(\alpha n+\beta)\,(n+1)!\,(n+1)!}$$

This proves (2.1.2).

By substitute $q=r=1$ in (2.1.2) we get (2.1.3).

Theorem 2.2. If $\alpha,\beta,\gamma,\delta,w\in C$; $Re(\alpha),Re(\beta),Re(\gamma),Re(\delta)>0$ and $q,r\in N$ then for $m\in N$,

$$\left(\frac{d}{dz}\right)^m E_{\alpha,\beta}^{\gamma,\delta,q,r}(z)=(\gamma)_{qm}(\delta)_{rm}\Sigma_{n=0}^{\infty}\frac{(\gamma+qm)_{qn}(\delta+rm)_{rn}z^n}{\Gamma(\alpha n+\beta+\alpha m)\,n!\,(n+m)!}\quad\text{... (2.2.1)}$$

$$\left(\frac{d}{dz}\right)^m\left[z^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta,q,r}(wz^\alpha)\right]=z^{\beta-m-1}E_{\alpha,\beta-m}^{\gamma,\delta,q,r}(wz^\alpha),\ Re(\beta-m)>0,\quad\text{... (2.2.2)}$$

In particular,

$$\left(\frac{d}{dz}\right)^m\left[z^{\beta-1}E_{\alpha,\beta}(wz^\alpha)\right]=z^{\beta-m-1}E_{\alpha,\beta-m}(wz^\alpha)\quad\text{... (2.2.3)}$$

And

$$\left(\frac{d}{dz}\right)^m\left[z^{\beta-1}\phi(\gamma,\beta;wz)\right]=\frac{\Gamma(\beta)}{\Gamma(\beta-m)}\left[z^{\beta-m-1}\phi(\gamma,\beta-m;wz)\right]\quad\text{... (2.2.4)}$$

Proof: From (1.5) we have

$$\left(\frac{d}{dz}\right)^m E_{\alpha,\beta}^{\gamma,\delta,q,r}(z)=\left(\frac{d}{dz}\right)^m\Sigma_{n=0}^{\infty}\frac{(\gamma)_{qn}(\delta)_{rn}z^n}{\Gamma(\alpha n+\beta)\,n!\,n!}$$

$$=\Sigma_{n=0}^{\infty}\frac{(\gamma)_{qn}(\delta)_{rn}n(n-1)(n-2)\ldots(n-m+1)z^{n-m}}{\Gamma(\alpha n+\beta)[n(n-1)(n-2)\ldots(n-m+1)]\,(n-m)!\,n!}$$

$$=\Sigma_{n=0}^{\infty}\frac{(\gamma)_{qn}(\delta)_{rn}z^{n-m}}{\Gamma(\alpha n+\beta)\,(n-m)!\,n!}$$

Put $n=n+m$ then

$$=\Sigma_{n=0}^{\infty}\frac{(\gamma)_{qn+qm}(\delta)_{rn+rm}z^n}{\Gamma(\alpha n+\beta+\alpha m)\,n!\,(n+m)!}$$

$$\left(\frac{d}{dz}\right)^m E_{\alpha,\beta}^{\gamma,\delta,q,r}(z)=(\gamma)_{qm}(\delta)_{rm}\Sigma_{n=0}^{\infty}\frac{(\gamma+qm)_{qn}(\delta+rm)_{rn}z^n}{\Gamma(\alpha n+\beta+\alpha m)\,n!\,(n+m)!}$$

This complete proves of (2.2.1).

$$\left(\frac{d}{dz}\right)^m\left[z^{\beta-1}\,E_{\alpha,\beta}^{\gamma,\delta,q,r}(wz^\alpha)\right]=\left(\frac{d}{dz}\right)^m\left[\Sigma_{n=0}^{\infty}\frac{(\gamma)_{qn}(\delta)_{rn}w^nz^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)\,n!\,n!}\right]$$

$$= \Sigma_{n=0}^{\infty} \frac{(\gamma)_{qn}(\delta)_{rn} w^n z^{\alpha n+\beta-1-m}\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta)\Gamma(\alpha n+\beta-m)\ n!\ n!}$$

$$= z^{\beta-m-1}\Sigma_{n=0}^{\infty} \frac{(\gamma)_{qn}(\delta)_{rn} w^n z^{\alpha n}}{\Gamma(\alpha n+\beta-m)\ n!\ n!}$$

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1}\, E_{\alpha,\beta}^{\gamma,\delta,q,r}(wz^{\alpha})\right] = z^{\beta-m-1}\, E_{\alpha,\beta-m}^{\gamma,\delta,q,r}(wz^{\alpha})$$

This complete prove of (2.2.2).

Let us consider $\gamma = \delta = q = r = 1$ in (2.2.2) then we get

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1}E_{\alpha,\beta}(wz^{\alpha})\right] = z^{\beta-m-1}E_{\alpha,\beta-m}(wz^{\alpha})$$

This complete prove of (2.2.3).

For the proof of (2.2.4) we take $\alpha = \ \ q = r = 1$ in (2.2.2) we get

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1}\phi(\gamma,\delta;\beta;wz)\right] = \frac{\Gamma(\beta)}{\Gamma(\beta-m)}\left[z^{\beta-m-1}\phi(\gamma,\delta;\beta-m;wz)\right]$$

This complete the proof of (2.2.4).

Corollary 2.3. If $\alpha,\beta,\gamma \in C$, $Re(\alpha) > 0$; $q \in N$ then

$$E_{\alpha,\beta}^{\gamma,q}(z) = \beta E_{\alpha,\beta+1}^{\gamma,q}(z) + \alpha z\frac{d}{dz}E_{\alpha,\beta+1}^{\gamma,q}(z), \qquad \ldots(2.3.1)$$

$$E_{\alpha,\beta-\alpha}^{\gamma,q}(z) - E_{\alpha,\beta-\alpha}^{\gamma-1,,q}(z) = qz\Sigma_{n=0}^{\infty}\frac{(\gamma)_{qn+q-1}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!.}, \qquad \ldots(2.3.2)$$

in particular,

$$E_{\alpha,\beta-\alpha}^{\gamma,q}(z) - E_{\alpha,\beta-\alpha}^{\gamma-1,q}(z) = z\,E_{\alpha,\beta}^{\gamma,q}(z). \qquad \ldots(2.3.3)$$

Proof: If we take $\delta = 1, r = 1$ in (2.1.1), (2.1.2) and (2.1.3) then we get the result.

Corollary 2.4. If $\alpha,\beta,\gamma \in C$, $Re(\alpha) > 0$; then

$$E_{\alpha,\beta}^{\gamma}(z) = \beta E_{\alpha,\beta+1}^{\gamma}(z) + \alpha z\frac{d}{dz}E_{\alpha,\beta+1}^{\gamma}(z), \qquad \ldots(2.4.1)$$

$$E_{\alpha,\beta-\alpha}^{\gamma,q}(z) - E_{\alpha,\beta-\alpha}^{\gamma-1,}(z) = z\Sigma_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!.}, \qquad \ldots(2.4.2)$$

in particular,

$$E_{\alpha,\beta-\alpha}^{\gamma}(z) - E_{\alpha,\beta-\alpha}^{\gamma-1}(z) = z\,E_{\alpha,\beta}^{\gamma}(z). \qquad \ldots(2.4.3)$$

Proof: If we take $\delta = 1, q = \ r = 1$ in (2.1.1), (2.1.2) and (2.1.3) then we get the result.

Corollary 2.5. If $\alpha,\beta \in C$, $Re(\alpha) > 0$; then

$$E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z\frac{d}{dz}E_{\alpha,\beta+1}(z), \qquad \ldots(2.5.1)$$

$$E_{\alpha,\beta-\alpha}(z) - E_{\alpha,\beta-\alpha}(z) = qz\Sigma_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \quad \text{... (2.5.2)}$$

in particular,

$$E_{\alpha,\beta-\alpha}^{\gamma,q}(z) - E_{\alpha,\beta-\alpha}^{\gamma-1,q}(z) = z\, E_{\alpha,\beta}^{\gamma,q}(z). \quad \text{... (2.5.3)}$$

Proof: If we take $\gamma = \delta = 1, q = r = 1$ in (2.1.1), (2.1.2) and (2.1.3) then we get the result.

Corollary 2.6. If $\alpha, \beta, \gamma, w \in C$; $Re(\alpha), Re(\beta), Re(\gamma) > 0$ and $q \in N$ then for $m \in N$,

$$\left(\frac{d}{dz}\right)^m E_{\alpha,\beta}^{\gamma,q}(z) = (\gamma)_{qm} E_{\alpha,\beta+m\alpha}^{\gamma+qm,q,}(z) = (\delta)_{rm} E_{\alpha,\beta+m\alpha}^{\gamma+qm,,q}(z) \quad \text{... (2.6.1)}$$

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E_{\alpha,\beta}^{\gamma,q}(wz^{\alpha})\right] = z^{\beta-m-1} E_{\alpha,\beta-m}^{\gamma,q}(wz^{\alpha}), Re(\beta-m) > 0, \quad \text{... (2.6.2)}$$

Proof: If we take $\delta = 1, r = 1$ in (2.2.1) and (2.2.2) then we get the result.

GENERALIZED HYPERGEOMETRIC FUNCTION REPRESENTATION OF $E_{\alpha,\beta}^{\gamma,\delta,q,r}(z)$.

Using (1.5), taking $\alpha = k \in N$, and $q, r \in N$ we have

$$E_{\alpha,\beta}^{\gamma,\delta,q,r}(z) = \Sigma_{n=0}^{\infty} \frac{(\gamma)_{qn}(\delta)_{rn} z^n}{\Gamma(\alpha n+\beta)\, n!\, n!}$$

$$E_{k,\beta}^{\gamma,\delta,q,r}(z) = \frac{1}{\Gamma(\beta)} \Sigma_{n=0}^{\infty} \frac{(\gamma)_{qn}(\delta)_{rn} z^n}{(\beta)_{kn}\, (1)_n\, n!}$$

$$E_{k,\beta}^{\gamma,\delta,q,r}(z) = \frac{1}{\Gamma(\beta)} \Sigma_{n=0}^{\infty} \frac{\prod_{i=1}^{q}\left(\frac{\gamma+i-1}{q}\right)_n \prod_{j=1}^{r}\left(\frac{\delta+j-1}{r}\right)_n \left(\frac{q^q r^r z}{k^{2k}}\right)^n}{\prod_{l=1}^{k}\left(\frac{\beta+l-1}{k}\right)_n (1)_n\, n!}$$

$$E_{k,\beta}^{\gamma,\delta,q,r}(z) = \frac{1}{\Gamma(\beta)} F_{k+1}^{q+r}\left[\begin{matrix}\left(\frac{\gamma}{q}\right), \left(\frac{\gamma+1}{q}\right), \dots, \left(\frac{\gamma+q-1}{q}\right), \left(\frac{\delta}{r}\right), \left(\frac{\delta+1}{r}\right), \dots, \left(\frac{\delta+r-1}{r}\right) \\ \left(\frac{\beta}{k}\right), \left(\frac{\beta+1}{k}\right), \dots, \left(\frac{\beta+k-1}{k}\right), 1\end{matrix} \,\middle|\, \frac{r^r q^q z}{k^{2k}}\right]$$

Convergence criteria for generalized hypergeometric function F_{k+1}^{q+r} :

(i) If $q + r < k$, the function F_{k+1}^{q+r} converges for all finite z.

(ii) If $q + r = k + 1$, the function F_{k+1}^{q+r} converges for $|z| < 1$ and diverges for $|z| > 1$.

(iii) If $q + r > k + 1$, the function F_{k+1}^{q+r} is divergent for $z \neq 0$.

(iv) If $q + r = k + 1$, the function F_{k+1}^{q+r} is absolutely convergent on the circle $|z| = 1$ if

$$\mathrm{Re}\left(\Sigma_{l=1}^{k}\left(\frac{\beta+l-1}{k}\right)-\Sigma_{j=1}^{q}\left(\frac{\gamma+j-1}{q}\right)-\Sigma_{j=1}^{r}\left(\frac{\delta+j-1}{r}\right)\right)>0.$$

MELLIN–BARNES INTEGRAL REPRESENTATION OF $E_{\alpha,\beta}^{\gamma,\delta,q,r}(z)$.

Theorem 4.1. Let $\alpha \in R+$; γ, $\delta \in C$ ($\gamma = 0$) and $q \in N$. Then the function $E_{\alpha,\beta}^{\gamma,\delta,q,r}(z)$ is represented by the Mellin–Barnes integral as

$$E_{\alpha,\beta}^{\gamma,\delta,q,r}(z)=\frac{1}{2\pi i\Gamma\gamma\Gamma\delta}\int_{L}\frac{\Gamma(s)\Gamma(\gamma-qs)\Gamma(\delta-rs)}{\Gamma(\beta-\alpha s)}(-z)^{-s}ds \qquad \ldots(4.1.1)$$

where $|\arg(z)| < \pi$; the contour of integration beginning at $-i\infty$ and ending at $+i\infty$, and indented to separate the poles of the integrand at $s = -n$ for all $n \in N_0$ (to the left) from those at $s = \frac{\gamma+n}{q}$ for all $n \in N_0$ (to the right).

Proof. We shall evaluate the integral on the RHS of (4.1.1) as the sum of the residues at the poles $s = 0, -1, -2, \ldots$. We have

$$=\frac{1}{2\pi i}\int_{L}\frac{\Gamma(s)\Gamma(\gamma-qs)\Gamma(\delta-rs)}{\Gamma(\beta-\alpha s)}(-z)^{-s}ds$$

$$=\Sigma_{n=0}^{\infty}\mathrm{Re}_{s=-n}\, s\left[\frac{\Gamma(s)\Gamma(\gamma-qs)\Gamma(\delta-rs)}{\Gamma(\beta-\alpha s)}(-z)^{-s}\right]$$

$$=\Sigma_{n=0}^{\infty}\lim_{s\to -n}\frac{\pi(s+n)}{\sin\pi s}\,\frac{1}{\Gamma(1-s)}\,\frac{\Gamma(\gamma-qs)\Gamma(\delta-rs)}{\Gamma(\beta-\alpha s)}(-z)^{-s}$$

$$=\Sigma_{n=0}^{\infty}(-1)^{n}\frac{1}{\Gamma(1+n)}\,\frac{\Gamma(\gamma+qn)\Gamma(\delta+rn)}{\Gamma(\beta+\alpha n)}(-z)^{n}$$

$$=\Gamma(\gamma)\Gamma(\delta)\,E_{\alpha,\beta}^{\gamma,\delta,q,r}(z).$$

That is

$$E_{\alpha,\beta}^{\gamma,\delta,q,r}(z)=\frac{1}{2\pi i\Gamma\gamma\Gamma\delta}\int_{L}\frac{\Gamma(s)\Gamma(\gamma-qs)\Gamma(\delta-rs)}{\Gamma(\beta-\alpha s)}(-z)^{-s}ds.$$

RELATIONSHIP WITH SOME KNOWN SPECIAL FUNCTIONS (GENERALISED LAGUERRE POLYNOMIALS, FOX H-FUNCTION, WRIGHT HYPERGEOMETRIC FUNCTION)

5.1. Relationship with generalised Laguerre polynomials

Putting $\alpha = k, \beta = \mu+1$, $\gamma = -m$, $\delta = 1, r = 1$, $q \in N$ with $q \mid m$ and replacing z by zk in (1.5), we get

$$E_{k,\mu+1}^{-m,q,1,1}\left(z^k\right) = \Sigma_{n=0}^{\left[\frac{m}{q}\right]} \frac{(-m)_{qn}\, z^{kn}}{\Gamma(kn+\mu+1)n!}$$

$$= \Sigma_{n=0}^{\left[\frac{m}{q}\right]} \frac{(-1)^{qn} m!\ z^{kn}}{(m-qn)!\ \ \Gamma(kn+\mu+1)n!}$$

$$= \frac{\Gamma(m+1)}{\Gamma(km+\mu+1)} \Sigma_{n=0}^{\left[\frac{m}{q}\right]} \frac{(-1)^{qn}\Gamma(km+\mu+1)\, z^{kn}}{(m-qn)!\ \ \Gamma(kn+\mu+1)n!}$$

$$E_{k,\mu+1}^{-m,q,1,1}\left(z^k\right) = \frac{\Gamma(m+1)}{\Gamma(km+\mu+1)} Z_{\left[\frac{m}{q}\right]}^{(\mu)}(z,k), \qquad \ldots (5.1.1)$$

where $Z_{\left[\frac{m}{q}\right]}^{(\mu)}(z,k)$ is a polynomial of $\left[\frac{m}{q}\right]$ in z^k.

In particular, $Z_m^{(\mu)}(z,1) = L_m^{(\mu)}(z)$ so that

$$E_{k,\mu+1}^{-m}(z) = \frac{\Gamma(m+1)}{\Gamma(km+\mu+1)} L_m^{(\mu)}(z,1). \qquad \ldots (5.1.2)$$

5.2. Relationship with Fox H-function

Using (4.1.1), we get

$$E_{\alpha,\beta}^{\gamma,\delta,q,r}(z) = \frac{1}{2\pi i\Gamma\gamma\Gamma\delta} \int_L \frac{\Gamma(s)\Gamma(\gamma-qs)\Gamma(\delta-rs)}{\Gamma(\beta-\alpha s)} (-z)^{-s} ds$$

$$= \frac{1}{\Gamma(\gamma)} H_{2,3}^{1,2}\left[-z\middle| \begin{matrix} ((1-\gamma),q),((1-\delta),r) \\ (0,1),((1-\beta),\alpha) \end{matrix}\right]. \qquad \ldots (5.2.1)$$

5.3. Relationship with Wright hypergeometric function

If $q, r \in (0,1)$ then (1.5) can be written as

$$E_{\alpha,\beta}^{\gamma,\delta,q,r}(z) = \frac{1}{\Gamma(\gamma)\Gamma(\delta)} \Sigma_{n=0}^{\infty} \frac{\Gamma(\gamma+qn)\Gamma(\delta+rn)\, z^n}{\Gamma(\alpha n+\beta)\,(1)_n\, n!}$$

$$E_{\alpha,\beta}^{\gamma,\delta,q,r}(z) = \frac{1}{\Gamma(\gamma)\Gamma(\delta)} \Psi_2^2\left[\begin{matrix}(\gamma,q),(\delta,r) \\ (\beta,\alpha),1\end{matrix}\middle| z\right]. \qquad \ldots (5.3.2)$$

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