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INTEGRAL SOLUTION OF THE HOMOGENEOUS TERNARY CUBIC EQUATION

$$x^3 + y^3 = 52(x+y)z^2$$

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The cubic homogeneous ternary equation $x^3 + y^3 = 52(x+y)z^2$ is analyzed for its non-zero integral solutions. A few interesting relations between the solutions and special numbers are exhibited.

Key-Words: Homogeneous, ternary, Diophantine equation, integral solution.

Notations: $P_{rn} = n(n+1)$

$$S_n = 6n(n-1) + 1$$

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

INTRODUCTION

Integral solutions for the homogeneous or non-homogeneous Diophantine equation is an interesting concept as it can be seen from [1-4]. In [5-13], a few special cases of cubic Diophantine equation with three and four unknowns are studied. This communication concerns with another interesting homogeneous cubic equation with three unknowns given by $x^3 + y^3 = 52(x+y)z^2$ for its integral solutions.A few interesting relations between the solutions are presented.

METHOD OF ANALYSIS

The cubic Diophantine equation with three unknowns to be solved for getting non-zero integral solution is,

$$x^3 + y^3 = 52(x+y)z^2 \quad ...(1)$$

on substituting the linear transformations

$$x = u + v, y = u - v \quad ...(2)$$

leads to $u^2 + 3v^2 = 52z^2 \quad ...(3)$

We employ different ways of solving (3) and thus different patterns of integer solutions to (1) are illustrated below.

Pattern : I

Assume $z = a^2 + 3b^2$ ...(4)

Write 52 as $2 = (7 + i\sqrt{3})(7 - i\sqrt{3})$ ...(5)

Substituting (4) and (5) in (3) and using the method of factorization, define,

$$(u + i\sqrt{3}v) = (7 + i\sqrt{3})(a + i\sqrt{3}b)^2 \quad ...(6)$$

Equating real and imaginary parts in the above equation ,we get,

$$\left.\begin{array}{l} u = 7a^2 - 21b^2 - 6ab \\ v = a^2 - 3b^2 + 14ab \end{array}\right\} \quad ...(7)$$

Substituting (7) in (2) ,the corresponding integer values of x, y, z satisfying (1) are obtained as

$$x = x(a.b) = 8a^2 - 24b^2 + 8ab$$
$$y = y(a.b) = 6a^2 - 18b^2 - 20ab$$
$$z = z(a, b) = a^2 + 3b^2$$

Properties:

$$x(a,1) + z(a,1) - t_{10,a} \equiv 1(\mathrm{mod}\,2).$$

$$x(a,1) + y(a,1) - t_{28,a} - t_{4,a} \equiv 0(\mathrm{mod}\,3).$$

$x(a,1) + t_{8,a} - 2t_{4,a} + 25$ is a perfect square.

Pattern : II

In (3), 52 can also be written as

$$52 = (5 + 3i\sqrt{3})(5 - 3i\sqrt{3}) \quad ... (8)$$

Substituting (4) and (8) in (3), and following the same procedure as in Pattern I,we get non-zero distinct integral solutions of (1) as,

$$x = x(a, b) = 8a^2 - 24b^2 - 8ab$$
$$y = y(a, b) = 2a^2 - 6b^2 - 28ab$$
$$z = z(a, b) = a^2 + 3b^2$$

Properties:

$$x(a,1) + y(a,1) - t_{22,a} \equiv 0(\mathrm{mod}\,5).$$

$$x(a,1) + z(a,1) - t_{20,a} \equiv 0(\mathrm{mod}\,3).$$

$$y(a,1)+z(a,1)-2t_{4,a}+199 \text{ is a perfect square.}$$

Pattern : III

Equation (3) can also be written as

$$u^2+3v^2=52z^2*1 \quad ...(9)$$

Write 1 as

$$1=\frac{(1+i\sqrt{3})(1-i\sqrt{3})}{2^2} \quad ...(10)$$

Substituting (8) and (10) in (9) using the method of factorization define,

$$(u+i\sqrt{3}v)=(5+i3\sqrt{3})(a+i\sqrt{3}b)^2\frac{(1+i\sqrt{3})}{2}$$

Equating real and imaginary parts, we get the values of u,v as

$$u=-2a^2+6b^2-24ab$$
$$v=4a^2-12b^2-4ab$$

Substituting the values of u and v in (2), we get the non-zero distinct integral solution of (1) as

$$x=x(a,b)=2a^2-6b^2-28ab$$
$$y=y(a,b)=-6a^2+18b^2-20ab$$
$$z=z(a,b)=a^2+3b^2$$

Properties:

$$x(a,1)+y(a,1)+z(a,1)+t_{8,a}\equiv 0(mod\ 3).$$
$$2[x(a,1)+z(a,1)+34a+6] \text{ is a nasty number.}$$
$$y(1,b)+z(1,b)-t_{34,b}-S_b+P_{rb}+6=0$$

Note : 1 Using (8) and (10) in (9) and using the same procedure as in Pattern. III, we get the different set of nonzero distinct integer solution of (1) as

$$x=x(a,b)=2a^2-6b^2-28ab$$
$$y=y(a,b)=-6a^2+18b^2-20ab$$
$$z=z(a,b)=a^2+3b^2$$

Properties:

$$x(a,1)+y(a,1)+z(a,1)+t_{8,a}\equiv 0(mod5)$$
$$2[x(a,1)+z(a,1)+34a+6] \text{ is a nasty number.}$$
$$y(1,b)+z(1,b)-21Pr_b+5\equiv 0(mod41)$$

Pattern . IV In the equation (9), 1 can also be written as

$$1=\frac{(1+4i\sqrt{3})(1-4i\sqrt{3})}{7^2} \quad ...(11)$$

Using (8) and (11) in (9), using the method of factorization, define

$$(u+i\sqrt{3}v)=(5+i3\sqrt{3})(a+i\sqrt{3}b)^2\frac{(1+4i\sqrt{3})}{7}$$

Equating real and imaginary parts, we get the values of u, v as

$$u=\frac{-31a^2+93b^2-138ab}{7}$$

$$v=\frac{23a^2-69b^2-62ab}{7}$$

Substituting the values of u and v in (2), assuming $a=7A, b=7B$ we get the non-zero distinct integral solution of (1) as

$$x(A,B)=-56A^2+168B^2-1400AB$$

$$y(A,B)=-378A^2+1134B^2-532AB$$

$$z(A,B)=49A^2+147B^2$$

Properties:

$$x(A,1)+t_{114,}\text{A}-t_{2914,}\text{A}+1456t_{4,}\text{A}\equiv 0(mod\,2)$$

$$z(A,1)-x(A,1)-t_{202,A}-t_{12,A}\equiv 0(mod\,3)$$

$z(A,1)+14P_{rA}+14t_{4,A}-146$ is a perfect square.

Note : 2 Using (5) and (11) in (9) and using the same procedure as in Pattern. IV and assuming $a=7A, b=7B$, we get the different set of non-zero distinct integer solution of (1) as

$$x(A,B)=168A^2-504B^2-1288AB$$

$$y(A,B)=-238A^2+714B^2-1148AB.$$

$$z(A,B)=49A^2+147B^2$$

Properties:

$$x(A,1)-y(A,1)-t_{284,A}+265t_{4,A}\equiv 0(mod2).$$

$$z(A,1)+y(A,1)-t_{2300,A}+1338t_{\$,A}\equiv 0(mod3)$$

$$x(A,1)-168t_{4,A}\equiv 0(mod2)$$

Pattern .V The equation (3) can also be written as

$$52z^2-u^2=3*v^2 \quad ...(12)$$

Write 3 as

$$3=(\sqrt{52}+7)(\sqrt{52}-7) \quad ...(13)$$

Assume

$$v = 52a^2 - b^2 = (\sqrt{52}a + b)(\sqrt{52}a - b) \quad ...(14)$$

Using (13) and (14) in (12) using the method of factorization define

$$(\sqrt{52}z + u) = (\sqrt{52} + 7)(\sqrt{52}a + b)^2$$

Equating rational and irrational parts, we get

$$z = 52a^2 + b^2 + 14ab \quad ... (15)$$

$$u = 364a^2 + 7b^2 + 104ab \quad ...(16)$$

Substituting (14) and (16) in (2), we get the values of x, y as

$$\left.\begin{array}{l} x = 416a^2 + 6b^2 + 104ab \\ y = 312a^2 + 8b^2 + 104ab \end{array}\right\} \quad ...(17)$$

Thus (15) and (17) represents non-zero distinct integer solutions of (1).

.$3[x(a,1) - y(a,1) - 96t_{4,2} + 2$ is a nasty number.

$y(1,b) + z(1,b) - 4b + 3$ is a perfect square.

$x(1,b) + y(1,b) + z(1,b) - 15Pr_b \equiv 0(mod3)$

Pattern : VI Write (3) as

$$u^2 - 25z^2 = 27z^2 - 3v^2$$

$$\Rightarrow \quad (u + 5z)(u - 5z) = 3(3z + v)(3z - v) \quad ...(18)$$

Case : I

(18) can be written in the form of ratio as

$$\frac{(u + 5z)}{(3z + v)} = \frac{3(3z - v)}{(u - 5z)} = \frac{A}{B}$$

This is equivalent to the following system of equations as

$$uB + z(5B - 3A) - vA = 0$$

$$-Au + z(9B + 5A) - 3vB = 0$$

solving these two equations using cross multiplication method, we get the values of u, v and z as

$$u = 5A^2 + 18AB - 15A^2$$

$$v = -3A^2 + 10AB + 9B^2$$

$$z = A^2 + 3B^2$$

Substituting the values of u, v in (2), the non-zero distinct integral values satisfying (1) are obtained as

$$x = x(A,B) = 2A^2 + 28AB - 6B^2$$

$$y = y(A,B) = 8A^2 + 8AB - 24B^2$$
$$z = z(A,B) = A^2 + 3B^2$$

Properties:

$$x(A,1) + y(A,1) - t_{4,A} \equiv 0(\text{mod}2)$$
$$y(A,1) + z(A,1) - 9Pr_A - t_{3,A} + t_{4,A} + 21 = 0$$
$$x(1,B) + y(1,B) + z(1,B) - 36Pr_B + 63t_{4,B} = 11$$

Case . 2 (18) can also be written in the form of ratio as

$$\frac{(u-5z)}{3(3z-v)} = \frac{(3z+v)}{(u+5z)} = \frac{A}{B}$$

which is equivalent to the system of equations as

$$uB - z(5B + 9A) + 3vA = 0$$
$$\boldsymbol{-uA + z(3B - 5A) + vB = 0}$$

solving these two equations using cross multiplication method ,we get the values of u, v and z as

$$u = 15A^2 - 18AB - 5B^2$$
$$v = -9A^2 - 10AB + 3B^2$$
$$z = -3A^2 - B^2$$

Substituting the values of u, v in (2), we get the non-zero distinct integral solutions of (1) as

$$x = x(A,B) = 6A^2 - 28AB - 2B^2$$
$$y = y(A,B) = 24A^2 - 8AB - 8B^2$$
$$z = z(A,B) = -3A^2 - B^2$$

Properties:

$x(A,1) + y(A,1) + 6t_{4,A} + 19$ is a perfect square.

$2[y(A,1) - z(A,1) - 10Pr_A + 10t_{4,A} + 10]$ is a nasty number

$z(1,B) - x(1,B) + 25t_{4,B} - 28Pr_B + 3 = 0$

Case : 3 Write the equation (18) in the form of ratio as

$$\frac{(u+5z)}{(3z-v)} = \frac{3(3z+v)}{(u-5z)} = \frac{A}{B}$$

which is equivalent to the system of double equations as

$$uB + z(5B - 3A) + Av = 0$$
$$-uA + z(9B + 5A) + 3vB = 0$$

Solving these two equations using cross multiplication method we get the values of u, v and z as

$$u = -5A^2 - 18AB + 15B^2$$
$$v = -3A^2 + 10AB + 9B^2$$
$$z = -A^2 - 3B^2$$

Substituting the values of u, v in (2),the non-zero distinct integral values satisfying (1) are obtained as

$$x = x(A, B) = -8A^2 - 8AB + 24B^2$$
$$y = y(A, B) = -2A^2 - 28AB + 6B^2$$
$$z = z(A, B) = -A^2 - 3B^2$$

Properties :

$$x(A, 1) + y(A, 1) + z(A, 1) + 11t_{4,A} \equiv 0(mod3)$$
$$y(1, B) + z(1, B) + 28Pr_B - 31t_{74,B} + 3 = 0$$
$$x(1, B) + y(1, B) + z(1, B) + 9t_{4,B} - t_{74,B} \equiv 0(mod11)$$

Case;4 (18) can also be written in the form of ratio as

$$\frac{(u-5z)}{3(3z+v)} = \frac{(3z-v)}{(u+5z)} = \frac{A}{B}$$

which is equivalent to the system of double equations as

$$uB - z(5B + 9A) - 3Av = 0$$
$$-uA + z(3B - 5A) - vB = 0$$

Solving these two equations using cross multiplication method we get the values of u, v and z as

$$u = -15A^2 + 18AB - 5B^2$$
$$v = -9A^2 - 10AB + 3B^2$$
$$z = 3A^2 + B^2$$

Substituting the values of u, v in (2), the non-zero distinct integral values satisfying (1) are obtained as

$$x = x(A, B) = -24A^2 + 8AB - 2B^2$$
$$y = y(A, B) = -6A^2 + 28AB - 8B^2$$
$$z = z(A, B) = 3A^2 + B^2$$

Properties:

$$x(A,1) + y(A,1) + 66t_{4,A} - 36\,\mathrm{Pr}_A + 10 = 0$$
$$z(A, A+1) - y(A, A+1) + 10\,\mathrm{Pr}_A = 9$$

$$y(1,B)+z(1,B)+35t_{4,B}-28\Pr_B+3=0$$

Conclusion

In this paper, a search is performed to obtain different sets of non-zero integral solutions to the homogeneous ternary equation (1).One may search for other choices of integer solutions and their corresponding properties.

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