Acta Ciencia Indica, Vol. XLVII-M, No. 1 to 4, 1 (2021)

R-GENERALIZED AND SPECIAL R-GENERALIZED RECURRENT FINSLER SPACE

ANKIT MAURYA

Singhania University, Pacheri Bari (Jhunjhunu), Rajasthan, India

JITENDRA SINGH

Department of Physics, Sri. Lal Bahadur Shastri Degree College Gonda, Uttar Pradesh, India

RECEIVED : 24 July, 2021

This paper has been devoted to the study of R-generalized and special R-generalized recurrent Finsler spaces of the second order (of first and second kind). In this paper first section is introductory. In the second section we have studied curvature tensors arising from Berwald connection and Cartan connection. In the third section we have taken into account Cartan's third curvature tensor R^i_{jkh} and have defined *R*- generalized recurrent Finsler space of the second order (of first and second kinds) also we have defined special *R*-generalized recurrent Finsler spaces of the second order (of first and second kinds). During the course of these studies we have derived the some conditions under which *R*-generalized recurrent Finsler spaces and special *R*-generalized recurrent Finsler space of the second order of first and second kinds. In this paper we have also established the characterizing properties of generalized *R*- recurrent Finsler space and special R-generalized recurrent Finsler space of the second order and also we have established the conditions which Ricci tensor R_{jk} satisfies in such first and second kinds of recurrent spaces.

Keywords: *R*- generalized and special *R*-generalized recurrent Finsler spaces, Cartan's third curvature tensor, Ricci tensor R_{jk}.

INTRODUCTION

Matsumoto [1], Walker [11], Adati and Miyazawa [5] have studied the properties of recurrent Riemannian space by assuming different types of curvature tensors in V_n. Mishra and Pande [2], Sen, R.N.[9] have studied the recurrent Finsler spaces, Chaki and Roy Chandham [3] have introduced the Ricci-recurrent spaces of the second order in Riemannian geometry. Sinha and Singh [7] have discussed the recurrent Finsler spaces of the second order and have studied the properties of the recurrence vector and tensor fields, they have also discussed the recurrence properties of the Berwald's curvature tensor field H^i_{jkh} equipped with symmetric connection coefficients G^i_{jk} in an n-dimensional Finsler space. Roy [8] has defined genralised 2- recurrent Remannian space. An attempt to extend the theory of genralised 2-recurrent curvature tensor to Finsler geometry has been made by Pande and Khan [4]. Mishra [6] has given a comparative study of various type of recurrent Finsler space by using Barwald's and Cartan's curvature tensors.

CURVATURE TENSORS ARISING FROM BERWALD CONNECTION AND CARTAN CONNECTION:

The geodesic deviation has been given in the following form: (Rund [13])

$$\frac{\delta^2 Z^j}{\delta u^2} + H^j_k(x,\dot{x})\, Z^k = 0, \qquad ...(2.1)$$

where Z^k is called the "variation vector" and the tensor $H^j_k\ (x,\dot{x})$ is said to be the deviation tensor defined by

$$H^j_k\ (x,\dot{x}) = K^j_{ihk}\dot{x}^i\dot{x}^h\ . \qquad ...(2.2)$$

It can be also written in the form

$$H^i_k = 2\partial_k G^i - \partial_h G^i_k \dot{x}^h + 2G^i_{ks}G^s - \dot{\partial}_s G^i \partial_k G^s\ , \qquad ...(2.3)$$

where, we have using of the fact that the function $G^s(x,\dot{x})$ is positively homogeneous of degree two in $\dot{x}^i$'s. The tensor defined by

(a) $H^i_{jk}\ (x,\dot{x}) = -\frac{1}{3}\ (\dot{\partial}_j H^i_k - \dot{\partial}_k H^i_j)$ ...(2.4)

and (b) $H^i_{jks}\ (x,\dot{x}) = \dot{\partial}_j H^i_{ks} = -\frac{1}{3}\dot{\partial}_j(\dot{\partial}_k H^i_s - \dot{\partial}_S H^i_k)$

are given in terms of $G^i(x,\dot{x})$ by the following equations:

$$H^i_{jk}\ (x,\dot{x}) = \partial_k\dot{\partial}_j G^i - \partial_j\dot{\partial}_k G^i + G^i_{kt}\dot{\partial}_j G^t - G^i_{tj}\dot{\partial}_k G^t, \qquad ...(2.5)$$

and

$$H^i_{hjk}\ (x,\dot{x}) = \partial_k G^i_{hj} - \partial_j G^i_{hk} + G^t_{hj}G^i_{tk} - G^t_{hk}G^i_{tj} + G^i_{thk}\dot{\partial}_j G^t - G^i_{thj}\dot{\partial}_k G^t, \qquad ...(2.6)$$

respectively, where

(a) $G^i_{hjk}\ (x,\dot{x}) \stackrel{\text{def}}{=} \dot{\partial}_h G^i_{jk}$ ...(2.7)

(b) $G^i_{hjk}\dot{x}^h = 0.$

The Curvature tensor H^i_{hjk} and K^i_{hjk} are related by

$$H^i_{hjk} = K^i_{hjk} + \dot{x}^r\ \dot{\partial}_h K^i_{rjk}\ , \qquad ...(2.8)$$

$$H_{ijks}\dot{x}^i = K_{ijks}\dot{x}^i \qquad ...(2.9)$$

and $$H^i_{hjk}\,\dot{x}^h = K^i_{hjk}\,\dot{x}^h = H^i_{jk} \;. \qquad ...(2.10)$$

The Berwald curvature tensor satisfies the following identities:

$$H^i_{hjk} + H^i_{jkh} + H^i_{khj} = 0 \;, \qquad ...(2.11)$$

$$\{H^r_{sik(t)} + H^r_{sti(k)} + H^r_{skt(i)}\}\,\dot{x}^s = 0, \qquad ...(2.12)$$

$$H^r_{ik(t)} + H^r_{ti(k)} + H^r_{kt(i)} = 0, \qquad ...(2.13)$$

$$H^r_{jik(t)} + H^r_{jti(k)} + H^r_{jkt(i)} + H^m_{ik}\,G^r_{mjt} + H^m_{ti}\,G^r_{mjk} + H^m_{kt}\,G^r_{mji} = 0. \qquad ...(2.14)$$

The following contractions shall be used in the following form:

(a) $H_i = H^h_{ih}$ ...(2.15)

(b) $H_{ij} = H^h_{ijh} = \dot{\partial}_i H_j$,

(c) $H_{jh} - H_{hj} = H^s_{shj}$,

(d) $H^i_{hj}\,\dot{x}^h = H^i_j$,

(e) $H^i_j\,\dot{x}^j = 0$,

(f) $\dot{\partial}_i H^i_j\,\dot{x}^j = -H^i_i$,

(g) $H^i_i = (n-1)H$.

The commutation formulae involving the tensor H^i_{jkt} and G^i_{jkt} are given by

(a) $T_{(h)(k)} - T_{(k)(h)} = (-\partial_i T)\,H^i_{hk}$, ...(2.16)

(b) $T^i_{j(h)(k)} - T^i_{j(k)(h)} = -\dot{\partial}_r T^i_j H^r_{hk} + T^r_j H^i_{rhk} - T^i_r H^r_{jhk}$

and (a) $(\dot{\partial}_k T)_{(l)} - \dot{\partial}_k (T_{(l)}) = 0$, ...(2.17)

(b) $(\dot{\partial}_k T^i_j)_{(h)} - \dot{\partial}_k (T^i_{j(h)}) = T^i_r G^r_{jkh} - T^r_j G^i_{rkh}$.

The Ricci identities for a tensor T^i_j involving h- and v- covariant derivative with respect to the Cartan connection are given by [11]

$$T^i_{j|h|k} - T^i_{j|k|h} = T^m_j R^i_{mhk} - T^i_m R^m_{jhk} - T^i_{j|m} R^m_{hk} \;, \qquad ...(2.18)$$

$$T^i_{j|h|k} - T^i_{j|k|h} = T^m_j P^i_{mhk} - T^i_m P^m_{jhk} - T^i_{j|m} C^m_{hk} - T^i_{j|m} P^m_{hk} \;, \qquad ...(2.19)$$

$$T^i_{j|h|k} - T^i_{j|k|h} = T^m_j S^i_{mhk} - T^i_m S^m_{jhk} \qquad ...(2.20)$$

where $$R^i_{hkm} = \Theta_{(hm)},\left\{\frac{\partial \Gamma^{*i}_{hk}}{\partial x^m} - \frac{\partial \Gamma^{*i}_{hk}}{\partial \dot{x}^r}\,G^r_m + \Gamma^{*r}_{hk}\Gamma^{*i}_{rm} + C^i_{hr} R^r_{km}\right\} \qquad ...(2.21)$$

$$R^i_{hk} = R^i_{hkm}\,\dot{x}^m = \Theta_{(hm)}\left\{\frac{\partial G^i_h}{\partial x^k} - \frac{\partial G^i_h}{\partial \dot{x}^m}\,G^m_k\right\}, \qquad ...(2.22)$$

The tensors defined by (2.21), (2.22) and (2.23) are called Cartan's curvature tensors. Also, it is known as h- curvature tensor, hv- curvature tensor and v- curvature tensor.

R-GENERALIZED AND SPECIAL R-GENERALIZED RECURRENT FINSLER SPACES OF THE SECOND ORDER :

In a Finsler space the Cartan's third curvature tensor $R^i_{jkh}(x,\dot{x})$ satisfies the recurrence properties with respect to the requisite connection Verma [12]. Therefore, the characterized property is given by

$$R^i_{jkh|m} = \mu_m \; R^i_{jkh} \;, \qquad ...(3.1)$$

where $\mu_m(x)$ is known as non-zero recurrence vector field. A *R*-recurrent space of order two is a Finsler space in which the curvature tensor under assumption (with respect to the Cartan's connection Γ^{*i}_{jk}) satisfies

$$R^i_{jkh|m|l} = \alpha_{lm} \; R^i_{jkh} \;, \qquad ...(3.2)$$

where $\alpha_{lm}(x,\dot{x})$ is associated tensor of recurrence and it is covariant of order two.

We now take into consideration a Finsler space in which the curvature under assumption satisfies

$$R^i_{jkh|m|l} = \mu_l \; R^i_{jkh|m} + \alpha_{lm} \; R^i_{jkh} \qquad ...(3.3)$$

and

$$R^i_{jkh|m|l} = \mu_m \; R^i_{jkh|l} + \alpha_{lm} \; R^i_{jkh} \;. \qquad ...(3.4)$$

Here, we shall call the space satisfying (3.3) and (3.4) as generalized *R*- recurrent Finsler space of the second order of first and second kinds respectively. In this continuation it has also been observed that if the curvature tensor under assumption satisfies

$$R^i_{jkh|m|l} = \mu_l \; R^i_{jkh|m} \qquad ...(3.5)$$

and

$$R^i_{jkh|m|l} = \mu_m \; R^i_{jkh|l} \;, \qquad ...(3.6)$$

then we call such a space as special *R*- generalized recurrent Finsler space of the second order of first and second kinds respectively. If we further suppose that the recurrence vector appearing in (3.3) and (3.4) is zero then the space reduces into a *R*- recurrent Finsler space of the second order. We now transvect the equations (3.3) to (3.6) by covariant fundamental tensor g_{ip} and thereafter, we get

$$R_{pjkh|m|l} = \mu_l \; R_{pjkh|m} + \alpha_{lm} \; R_{pjkh} \;, \qquad ...(3.7)$$

$$R_{pjkh|m|l} = \mu_m \; R_{pjkh|l} + \alpha_{lm} \; R_{pjkh} \;, \qquad ...(3.8)$$

$$R_{pjkh|m|l} = \mu_l \; R_{pjkh|m} \;, \qquad ...(3.9)$$

and

$$R_{pjkh|m|l} = \mu_m \; R_{pjkh|l} \;, \qquad ...(3.10)$$

whereas, if we transvect the equations (3.7) to (3.10) by contravariant fundamental tensor g^{ip} immediately we get back the set of equations (3.3) to (3.6). Therefore we can state:

Theorem (3.1):

The characterizing properties of *R*-generalized and special *R*-generalized recurrent Finsler space of the second order of the two kinds are respectively given by the equation (3.7) to (3.10).

Theorem (3.2):

The equations given by (3.3) to (3.6) and (3.7) to (3.10) are respectively found to be equivalent in *R*- generalized and special *R*- generalized recurrent Finsler spaces of the second order of the first and second kinds.

Now allowing a contraction in the set of equations (3.3) to (3.6) with respect to the indices *i* and *h*, we get

$$R_{jk\,|m|l} = \mu_l\ R_{jk\,|m} + \alpha_{lm}\ R_{jk}\ , \quad \text{... (3.11)}$$

$$R_{jk\,|m|l} = \mu_m\ R_{jk\,|l} + \alpha_{lm}\ R_{jk}\ , \quad \text{...(3.12)}$$

$$R_{jk\,|m|l} = \mu_l\ R_{jk\,|m} \quad \text{...(3.13)}$$

and

$$R_{jk\,|m|l} = \mu_m\ R_{jk\,|l}\ , \quad \text{...(3.14)}$$

respectively. After these observations, therefore we can state:

Theorem (3.3):

The set of equations given by (3.11) to (3.14) are always satisfied by the Ricci tensor R_{jk} in a *R*- generalized and special *R*- generalized recurrent Finsler spaces of the second of the first and second kinds.

Now, here it can easily we observed that if the Ricci tensor satisfies (3.11) or (3.12) then it is not necessary that such a Ricci tensor be generalized *R*- recurrent of the second order of the two kinds. Also will arise the Ricci tensor satisfies (3.13) or (3.14), in such a case it is not necessary that Ricci tensor will be special generalized *R*- recurrent of the second order of the two kinds.

Matsumoto [1] introduced a tensor $R_{ijk\,h}$ in the form

$$R_{ijk\,h} = g_{ik}\ L_{jh} + g_{jh}\ L_{ik} \quad (-\ \text{i/h}) \quad \text{...(3.15)}$$

where (a) $L_{ik} = \frac{1}{(n-2)}\,(R_{ik} - \frac{\gamma}{2}\,g_{ij})$, ...(3.16)

(b) $\gamma = \frac{1}{(n-1)}\,R_i^i$

And called the characterizing space as a *R*3- like Finsler space. Let us now consider an *R*3- like generalized Ricci and special generalized Ricci recurrent of the second order of the two kinds then after applying the process as have been applied previously, we can get,

$$R_{ijk\,h|m|l} = \mu_l\ R_{ijk\,h|m} + \alpha_{lm}\ R_{ijk\,h}\ , \quad \text{... (3.17)}$$

$$R_{ijk\,h|m|l} = \mu_m\ R_{ijk\,h|l} + \alpha_{lm}\ R_{ijk\,h}\ , \quad \text{... (3.18)}$$

$$R_{ijk\,h|m|l} = \mu_l \; R_{ijk\,h|m} \qquad ...(3.19)$$

and
$$R_{ijk\,h|m|l} = \mu_m \; R_{ijk\,h|l} \;. \qquad ...(3.20)$$

After these observations, therefore we can state:

Theorem (3.4):

In a Finsler space F_n , R- generalized and special R- generalized recurrent spaces of the second order of the two kinds are Ricci generalized and special Ricci generalized of the first and second kinds but not necessarily the converse.

Transvection of the equation (3.3) to (3.6) by $\dot{x}^j$ gives

$$H^i_{kh|m|l} = \mu_l \; H^i_{kh|m} + \alpha_{lm} H^i_{kh} \; , \qquad ... (3.21)$$

$$H^i_{kh|m|l} = \mu_m \; H^i_{kh|l} + \alpha_{lm} \; H^i_{kh} \; , \qquad ...(3.22)$$

$$H^i_{kh|m|l} = \mu_l \; H^i_{kh|m} \qquad ...(3.23)$$

and
$$H^i_{kh|m|l} = \mu_m \; H^i_{kh|l} \; . \qquad ...(3.24)$$

Again tranvection of the equations (3.21) to (3.24) by $\dot{x}^k$ gives

$$H^i_{h|m|l} = \mu_l \; H^i_{h|m} + \alpha_{lm} \; H^i_h \; , \qquad ...(3.25)$$

$$H^i_{h|m|l} = \mu_m \; H^i_{h|l} + \alpha_{lm} \; H^i_h \; , \qquad ...(3.26)$$

$$H^i_{h|m|l} = \mu_l \; H^i_{h|m} \qquad ...(3.27)$$

and
$$H^i_{h|m|l} = \mu_m \; H^i_{h|l} \; . \qquad ...(3.28)$$

Now allowing a contraction in (3.21), (3.22), (3.23) and (3.24) with respect to the indices i and h, we have

$$H_{|m|l} = \mu_l \; H_{|m} + \alpha_{lm} \; H \; , \qquad ...(3.29)$$

$$H_{|m|l} = \mu_m \; H_{|l} + \alpha_{lm} \; H \; , \qquad ...(3.30)$$

$$H_{|m|l} = \mu_l \; H_{|m} \qquad ...(3.31)$$

and
$$H_{|m|l} = \mu_m \; H_{|l} \; . \qquad ...(3.32)$$

After these observations, therefore we can state:

Theorem (3.5):

In Finsler space F_n , R- generalized and special R- generalized recurrent Finsler space of the second order of the first and second kinds the tensors H^i_{kh} , H^i_h , the vector H_k and the scalar H are also R- generalized and special R- generalized recurrent of the second order of the two kinds.

CONCLUSION

This paper has been divided into three sections of which the first section is introductory. In the second section we have studied curvature tensors arising from Berwald connection and Cartan connection. In the third section we have taken into account Cartan's third curvature tensor R^i_{jkh} and have defined *R*- generalized recurrent Finsler space of the second order (of first and second kinds) also we have defined special *R*-generalized recurrent Finsler spaces of the second order (of first and second kinds). During the course of these studies we have derived the some conditions under which *R*-generalized recurrent Finsler spaces and special *R*-generalized recurrent Finsler space of the second order of first and second kinds. In this continuation we have established results in the form of theorems. The characterizing properties of generalized *R*- recurrent Finsler space and special *R*-generalized recurrent Finsler space of the second order and also we have established the conditions which Ricci tensor R_{jk} satisfies in such first and second kinds of recurrent spaces.

REFERENCES

1. M, Matsumoto: On Riemannian spaces with recurrent projective curvature, *Tensor (N.S.)*, **19**, 11-17, (1968).
2. Mishra, R.S. and Pande, H.D. : Recurrent Finsler spaces, *Jour. of Indian Math. Soc.* (1-2), **32**, 17-22, (1968).
3. Chaki, M.C. and Roy Chaudhary, A.N. : On Ricci recurrent spaces of second order, *J. Ind. Math. Soc.* 2, **19**, 279-287, (1967).
4. Pande, H.D. and Khan, T.A. : On Generalized 2 – recurrent Berwald's curvature tensor field in Finsler spaces, *Acta Ciencia Indica*. **4**(1), 56-59, (1977).
5. Adati, T. and Miyazawa, T. : On a Riemannian space with recurrent Conformal curvature, *Tensor (N.S.)* **18**, 348-354, (1967).
6. Mishra, R.B. : On a Recurrent Finsler space, *Rev. Roum. Math. Pures et Appl. Toma*, **18**, No. 5, 701-712, (1973).
7. Sinha, B.B. and Singh, S.P. : Recurrent Finsler spaces of second order in a Finsler space, *The Yakohama Math.* J.(2), **79-85**, (1971).
8. Roy. A.K. : On generalized 2- recurrent tensor in a Riemannian space, *Acad. Roy. Belg. Bull. cl. Science* (5), **58**, 220-228, (1972).
9. Sen, R.N. : Finsler spaces of recurrent curvature, *Tensor (N.S.)* **19**, 291-299, (1968).
10. Prakash, N. : Some remarks on recurrent and Ricci recurrent spaces, *Ganita* 2, **12**, 1-16, (1961).
11. Walker, A.G. : On Ruse's spaces of recurrent curvature, *Proc. Lond. Math. Soc.*, **53**, 36-64, (1950).
12. Verma, R. : Some transformation in Finsler spaces, *D. Phil. thesis, University of Allahabad*, Allahabad, India, (1992).
13. Rund, H. : The differential geometry of Finsler spaces, *Springer-Verlag*., Berlin (1959).

❑