THERMAL CONVECTION IN A VISCO-ELASTIC WALTER'S (MODEL-B) FLUID IN HYDROMAGNETICS THROUGH A POROUS MEDIUM

SUDHIR KUMAR PUNDIR

Department of Mathematics, S.D. (P.G.) College, Muzaffarnagar, U.P.-251001, India
Email: skpundir05@yahoo.co.in

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In the present study, thermal convection of visco-elastic walter's (model-B) fluid in porous medium in hydromagnetics is considered. In this paper, we examined the nature of perturbation at the marginal state taking $P_r = 0$. We have established variational principle in term of Rayleigh number *R* and the solution of the problem gives the extremum value of *R* over all possible functions satisfying the boundary conditions. We have also discussed the stability of the fluid layers confined between free boundaries. We find the sufficient condition for stability of the system is $R < \left[\frac{-R_d I_1 + QI_3}{a^2(I_5 + a^2 I_6)}, \frac{I_1 + R_d F I_1 + QP_2 I_4}{a^2 P_1 I_4} \right]$. Fixing the values of parameters, *Q*, R_d, P_1, P_2, we determine the value of Rayleigh number *R* for frequency *P* and any value of the wave number *a*.

Keywords: Thermal convection, Walter's model-B fluid, hydromagnetics, porous medium.

INTRODUCTION

The theory of Bénard convection in viscous Newtonian fluid layer heated from below has been given by Chandrasekhar **[1].** The instability problem of hydro magnetic viscous fluid has been studied by several researchers in past few decades. Through the discussion of the thermal instability of Maxwellian fluid in presence of magnetic field, Bhatia and Steiner **[2]** found that magnetic field has a stabilizing effect on visco-elastic fluid in the same way as for Newtonian fluid. The effect of transverse periodic variation of the permeability on the heat transfer has been studied by Singh *et al* **[3]**. They also studied the free convective flow of viscous incompressible fluid through a highly porous medium bounded by a vertical porous plate. A problem, governed by a coupled non-linear system of partial differential equation has been studied by Sounalgekar *et al* **[4]**. He have studied free convection flow of an incompressible viscous dissipative fluid. Wilson and Rallison **[5]** illustrated the instability of channel flows of elastic liquids having continuously stratified properties. Jha **[6]** analyzed the effect of applied magnetic field on transient free convective flow in a vertical channel. Rangnathan and Govindarajan **[7]** have studied the stabilization and destabilization of channel flow by location of viscosity stratified fluid layer. Singh and Sharma **[8]** analyzed three-dimensional free convective flow and heat transfer through a porous medium with periodic permeability. Batia and Mathur **[9]** discussed instability of visco-elastic superposed fluids in a vertical magnetic field through porous media. The fluids have been considered to be Newtonian or visco-elastic in all the above studies. Rhyzhov *et al*. **[10]** have studied instabilities in boundary-layer flows on a curved surface. Blanchette *et al*. **[11]** analyzed the stability of a stratified fluid with a vertically moving sidewall and shows the stability of uniform stratified fluid bounded by a sidewall moving vertically with constant velocity.

To the best of our knowledge, thermal convection of visco-elastic Walter's (Model B) fluid in porous medium in hydromagnetics is uninvestigated so far. Therefore, we have discussed the hydromagnetic instability of visco-elastic fluid Walters' (Model B) in porous medium. It can be looked upon as the extension of thermal instability of visco-elastic fluid layer in porous medium discussed by Rani **[12]** and Alam, Pundir **[13]**.

CONSTITUTIVE EQUATIONS

We consider the stability of an incompressible finitely conducting visco-elastic fluid layer in a porous media in the presence of a vertical magnetic field. The fluid is taken to be statically non-homogenous confined between to horizontal boundaries and heated from below. Let T_0 and T_1 [with $T_1 < T_0$] denote the uniform temperature of the lower and upper boundaries.

The stationary state of the system whose stability we wish to examine is given by following solutions of the basic conservation laws of the fluid flows:

$$\boldsymbol{q} = (0, 0, 0), \quad \text{...(1)}$$

$$T = T_0 - \beta z;\ \beta = \frac{T_0 - T_1}{d} > 0, \quad \text{...(2)}$$

$$\rho = \rho_0 \left[e^{-\delta z} + \alpha (T_0 - T_1) \right], \quad \text{...(3)}$$

$$P = P_0 - gP_0 \left[\frac{1}{\delta}(1 - e^{\delta z}) + \frac{\alpha \beta z^2}{2} \right] \quad \text{...(4)}$$

and

$$\boldsymbol{H} = (0, 0, H_0). \quad \text{...(5)}$$

where β is the uniform adverse temperature gradient maintained between the boundaries.

The equations of motion of Walters' (Model B) fluid in the porous medium in the presence of magnetic field are:

$$\rho\left[\frac{\partial \boldsymbol{q}}{\partial t}+(\boldsymbol{q}.\nabla)\boldsymbol{q}\right]=-\nabla P+\boldsymbol{g}\rho\lambda-g\alpha\rho\theta+\frac{\mu}{4\pi}(\nabla\times\boldsymbol{H})\times\boldsymbol{H}-\frac{1}{k_1}\left(\mu-\mu'\frac{\partial}{\partial t}\right)\boldsymbol{q} \quad \text{... (6)}$$

where $\boldsymbol{g}(0, 0, -g)$ is the gravitational force per unit mass, P is pressure. $\boldsymbol{H}$ is the magnetic field and k_1 is the permeability of the porous medium.

The induction equations are,

$$\frac{\partial \boldsymbol{H}}{\partial t}=\nabla\times(\boldsymbol{q}\times\boldsymbol{H})+\nabla^2\boldsymbol{H}, \quad \text{... (7)}$$

and

$$\nabla.\boldsymbol{H}=0. \quad \text{... (8)}$$

The equation of continuity is

$$\nabla.\boldsymbol{q}=0. \quad \text{... (9)}$$

The energy equation is,

$$\frac{\partial T}{\partial t}+(q.\nabla)T=\kappa_T\nabla^2 T \quad \text{... (10)}$$

BASIC STATE AND PERTURBATION EQUATIONS

To analyze the stability of the fluid layer, we perturbed the basic state of the fluid given by **(1)** to **(5)**. Let the perturbed state of the fluid layer be given by,

$$\mathbf{q}=(u, v, w),\ T'=T_0-\beta z+\theta,\ \rho'=\rho_0\left[e^{-\delta z}+\frac{\partial\rho}{\rho_0}+\alpha(T_0-T-\theta)\right],$$

$$P'=P+\delta P \quad \text{and} \quad \boldsymbol{H}=(h_x, h_y, H_0+h_z),$$

where (u, v, w) Q, $\delta\rho$, δP and (h_x, h_y, h_z) are respectively, the perturbation is velocity, temperature, density, pressure and magnetic field. Substituting these variables in constitutive equations, we have the linearized perturbation equations as,

$$\rho\frac{\partial u}{\partial t}=\frac{\partial\delta P}{\partial x}-\frac{1}{k_1}\left(\mu-\mu'\frac{\partial}{\partial t}\right)\mu+\frac{\mu}{4\pi}\left[H_z\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)\right], \quad \text{... (11)}$$

$$\rho\frac{\partial v}{\partial t}=\frac{\mu}{4\pi}\left[\frac{\partial H_x}{\partial x}-\frac{\partial H_x}{\partial y}\right]-\frac{1}{k_1}\left(\mu-\mu'\frac{\partial}{\partial t}\right)v, \quad \text{... (12)}$$

$$\rho\frac{\partial w}{\partial t}=-\frac{\partial}{\partial z}\delta P-g\delta\rho-\frac{1}{k_1}\left(\mu-\mu'\frac{\partial}{\partial t}\right)w, \quad \text{...(13)}$$

$$\delta\rho=-\alpha\rho\theta, \quad \text{...(14)}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \quad \text{... (15)}$$

$$\frac{\partial H_x}{\partial t} = H_0 \frac{\partial u}{\partial z} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) h_x, \quad \text{... (16)}$$

$$\frac{\partial H_y}{\partial t} = H_0 \frac{\partial v}{\partial z} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) h_y, \quad \text{... (17)}$$

$$\frac{\partial H_z}{\partial t} = H_0 \frac{\partial w}{\partial z} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) h_z, \quad \text{... (18)}$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \quad \text{... (19)}$$

and

$$\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} = \kappa \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \theta. \quad \text{... (20)}$$

To discuss the stability of the fluid layer, we consider the perturbation to be two discuss the stability of the fluid layer, we consider the perturbation to be two-dimensional and therefore we can take the perturbation variables of the form,

$$f(x, y, z, t) = \sum_{k,n} \exp.[ik_x + nt] \quad \text{... (21)}$$

where $f(z)$ is some regular function of z, representing the perturbation variables $f(x, y, z, t)$. In this, k is the wave number of n the complex growth rate of the perturbation modes. We substitute the complex growth rate of the perturbation equations **(11)** to **(20)** and then solve the resultant equations simultaneously. We have

$$\rho n u = -ik\delta P + \frac{\mu H_0}{4\pi}[Dh_x - ikh_z] - \frac{1}{k_1}(\mu - \mu' n)u, \quad \text{... (22)}$$

$$\rho n v = \frac{\mu H_0}{4\pi} Dh_y - \frac{1}{k_1}(\mu - \mu' n)v, \quad \text{...(23)}$$

$$\rho n w = -D\delta P - g\delta\rho - \frac{1}{k_1}(\mu - \mu' n)w, \quad \text{...(24)}$$

$$\delta\rho = -\alpha\rho\theta, \quad \text{...(25)}$$

$$iku + Dw = 0 \quad \text{or,} \quad iku = -Dw, \quad \text{...(26)}$$

$$nh_x = H_0 Du + \eta(D^2 - k^2)h_x, \quad \text{... (27)}$$

$$nh_y = H_0 Dv + \eta(D^2 - k^2)h_y, \quad \text{... (28)}$$

$$nh_z = H_0 Dw + \eta(D^2 - k^2)h_z, \quad \text{...(29)}$$

$$ikh_x + Dh_z = 0 \quad \text{... (30)}$$

and $$n\theta-\kappa(D^2-k^2)\theta=\beta w. \quad \text{...(31)}$$

We substitute the values of h_x and $\delta\rho$ from the equations **(30)** and **(25)** into the equations **(22)** and **(24),** we get

$$ik\rho nu=k^2\delta P-\frac{\mu H_0}{4\pi}[D^2-k^2]h_z-\frac{1}{k_1}(\mu-\mu' n)iku \quad \text{...(32)}$$

and $$n\rho w=-D\delta P+g\alpha\rho\theta-\frac{1}{k_1}(\mu-\mu' n)w. \quad \text{... (33)}$$

We eliminate δP from the equations **(32)** and **(33)** by multiplying the equation **(33)** by k^2 and differentiating the equation and adding, we get

$$ik\rho nDu=k^2D\delta P-\frac{\mu H_0}{4\pi}[D^2-k^2]h_z-\frac{1}{\kappa}(\mu-\mu' n)ikDu$$

$$k^2n\rho w=-D\delta Pk^2+g\alpha\rho\theta k^2-\frac{1}{\kappa_1}(\mu-\mu' n)k^2w$$

$$n\rho[ikDu+k^2w]=k^2g\alpha\rho\theta+\frac{\mu H_0}{4\pi}(D^2-k^2)h_z-\frac{1}{k_1}(\mu-\mu' n)(ikDu+k^2w). \quad \text{...(34)}$$

Eliminating u between the above equations **(34)** and **(26)**, we get

$$\left[-n\rho-\frac{1}{k_1}(\mu-\mu' n)\right](D^2-k^2)w+HD(D^2-k^2)h_z=k^2g\alpha\rho\theta. \quad \text{...(35)}$$

Now we have to solve the equations **(35)**, **(29)** and **(31)** to determine the nature of the perturbations.

We now, non-dimensionalize the perturbation variables in the equations **(35)**, **(29)** and **(31)** by taking the following transformations and dropping the *for convenience in writing,

$$D^*=dD, a=kd, P=\frac{nd^2}{v}, u^*=Uv, w^*=Uw,\ \upsilon=\frac{\mu}{\rho}, P_1=\frac{\upsilon}{\eta}, Q=\frac{H_0^2d^2}{\rho\upsilon^2}$$

$$R_d=\frac{d^2}{k_1}, R=\frac{g\alpha\beta d^4}{k\upsilon}, \lambda^*=\frac{\lambda\upsilon}{d^2}, F=\frac{v'}{d^2} \quad \text{... (36)}$$

Where P_1 is the Prandtl number, P_2 is the magnetic Prandtl number, Q is the magnetic force number, R_d the porosity number, R the Rayleigh number, λ^* non-dimensional relaxation time and F elastic parameter.

Using the above transformations **(36)** in equations **(35), (29)** and **(31)**, these equations become

$$[-P-R_d(1-FP)](D^2-a^2)w+QD(D^2-a^2)h_z=k^2g\alpha\rho\theta\rho\upsilon d^2$$

$$[-P-R_d(1-FP)](D^2-a^2)w+QD(D^2-a^2)h_z=a^2R\theta, \quad \textbf{...(37)}$$

$$(D^2-a^2-P_1P)\theta=-w \quad \textbf{...(38)}$$

and

$$(D^2-a^2-P_2P)h_z=-Dw. \quad \textbf{...(39)}$$

SUFFICIENT CONDITION FOR THE STABILITY OF THE SYSTEM

To analyze the nature of the perturbation modes, we have to solve the eigen value problem consisting of equations **(37)** and **(39)** together with the boundary condition. Now, multiplying the equation **(37)** by w*(complex conjugate of w), integrating the resultant equation over the interval (0, 1), we get,

$$-\left[P+R_d(FP)-1)\right]\int_0^1 w^*(D^2-a^2)w\,dz$$

$$+\int_0^1 w^*D(D^2-a^2)h_z dz=a^2R\int_0^1 w^*\theta dz\,. \quad \textbf{...(40)}$$

We express $G=(D^2-a^2)w$ and evaluate the above integrals using boundary condition, we find that

$$I_1=\int_0^1\left[|Dw|^2+a^2|w|^2\right]dz \quad \textbf{....(41)}$$

$$I_2=\int_0^1 Dw^*(D^2-a^2)h_z dz. \quad \textbf{... (42)}$$

Now, we take the complex conjugate of equation **(39).** We thus have

$$(D^2-a^2-P_2P^*)h_z^*=-Dw^*.$$

Substituting this value of $-Dw^*$in **(41),** we have

$$I_2=\int_0^1(D^2-a^2)^2h_zh_z^*dz-P_2P^*\int_z^* h_z^*(D^2-a^2)h_z dz.$$

Let $M=(D^2-a^2)h_z$, then

$$I_2=I_3+P_2P^*I_4, \quad \textbf{... (43)}$$

where, $I_3=\int_0^1|M|^2\,dz$, and $I_4=\int_0^1|Dh_z|^2+a^2|h_z|^2]dz.$

Now taking the complex conjugate of equation **(38)** and substituting the value of w^*, we have $\int_0^1 w^*\theta dz=I_5+(a^2+P_1P^*)I_6$. **...(44)**

where, $I_5=\int_0^1|D\theta|^2dz$ and $I_6=\int_0^1|\theta|^2dz$.

Substituting the values of the integrals $\int_0^1 w^*\theta dz, I_1, I_2$ and I_3 into the equation **(40),** we find the dispersion relation as

$$[P + R_d(FP - 1)]I_1 + Q(I_3 + P_2P^* I_4) = a^2R[I_5 + (a^2 + P_1P^*)I_6] \quad ...(45)$$

Now, P is the complex growth rate of the perturbations and so we can express $P = P_r + iP_i$. and taking the real part of this equation, we have

$$P_r[I_1 + R_dFI_1 + QP_2I_4 - a^2RP_1I_6] = a^2R(I_5 + a^2I_6) + R_dI_1 - QI_3.$$

Now, if $a^2R(I_5 + a^2I_6) + R_dI_1 - QI_3 < 0$

or $$a^2R(I_5 + a^2I_6) < -R_dI_1 + QI_3.$$

Or $$R < \left[\frac{-R_dI_1 + QI_3}{a^2(I_5 + a^2I_6)}\right]. \quad ... \textbf{(46)}$$

and if, $$I_1 + R_dFI_1 + QP_2I_4 - a^2RP_1I_6 < 0.$$

$$a^2RP_1I_6 < I_1 + R_dFI_1 + QP_2I_4.$$

or $$R < \left[\frac{I_1 + R_dFI_1 + QP_2I_4}{a^2P_1I_6}\right]. \quad ...\textbf{(47)}$$

Combining these two conditions **(46)** and **(47),** we see that sufficient condition for the stability of system is

$$R < \left[\frac{-R_dI_1 + QI_3}{a^2(I_5 + a^2I_6)}, \frac{I_1 + R_dFI_1 + QP_2I_4}{a^2P_1I_6}\right].. \quad ...\textbf{(48)}$$

MARGINAL STATE OF THE SYSTEM

We now proceed to examine the nature of the perturbations at the marginal state. We therefore take $P_r = 0$ and so we can express $P = iP_i$, (P_i is real) substituting this in the equation **(45).** We get

$$[iP_i + R_d(FiP_i - 1)]I_1 + Q(I_3 - P_2iP_iI_4) = a^2R[I_5 + (a^2 - ip_iP_1)I_6] \quad ...\textbf{(49)}$$

Separating the real and imaginary parts of the above equations, we have the real part

$$R = \frac{-R_dI_1 + QI_3}{a^2(I_5 + a^2I_6)}. \quad ... \textbf{(50)}$$

And from the imaginary part, we have

$$P_i[I_1 + R_d FI_1 - QP_2 I_4 + a^2 RP_1 I_6] = 0 \quad \textbf{...(51)}$$

If $P_i \neq 0$, then $a^2 RP_1 I_6 = QP_2 I_4 - I_1 - R_d FI_1$. ... **(52)**

Eliminating R between the equations **(50)** and **(52)**, we get

$$QP_2 I_4 - I_1 - R_d FI_1 - \frac{P_1(-R_d I_1 + QI_3)}{a^2\left(\frac{I_5}{I_6} + a^2\right)} = 0. \quad \textbf{...(53)}$$

By fixing the values of non-dimensional parameter Q, R_d, P_1, P_2 for any value of a and F, we calculate the value of P_i from the equation **(53)** as the root of the equation. Then with these values of P_i and a, we can find the value of R from the equation **(50).** We thus have a neutral mode.

VARIATIONAL PRINCIPLE

In this section, we shall establish variational principle in terms of Rayleigh number for the solution of this problem. Proceeding as in the previous section, the equation **(45)** can be expressed as.

$$[P + R_d(FP - 1)]I_1 + Q(I_3 + P_2 P^* I_4) = a^2 R[I_5 + (a^2 + P_1 P^*)I_6].$$

Taking the complex conjugate of the above equation and adding into it, we get

$$R = \frac{I}{a^2 J}, \quad \textbf{...(54)}$$

where $I = [P_r + R_d(FP_r - 1)]I_1 + Q[I_3 + P_2 P_r I_4]$ and $J = I_5 + (a^2 + P_1 P_r)I_6$.

Let δR be the variation in R, when W is subjected to a small variation δW and δG, δF, $\Delta\theta$, dh_z be the variations in G, F, q and h_z, respectively, consistent with the boundary conditions, *i.e.*,

$$\delta W = 0,\ \delta\theta = 0,\ \delta h_z = 0,\ \delta G = 0,\ \delta F = 0 \text{ at } z = 0 \text{ and } 1. \quad \textbf{...(55)}$$

Now we have

$$\delta R = \frac{I}{a^2 J}(\delta I - a^2 R\delta J). \quad \textbf{... (56)}$$

Taking up the variation δI, we have

$$2\delta I = [P + R_d(FP - 1) + P^* + R_d(FP^* - 1)\delta I_1 + Q\,[\delta I_3 + P_2(P + P^*)I_4] \quad \textbf{...(57)}$$

and $$2\delta J = 2\delta I_5 + (a^2 + P_1 P + a_2 + P_1 P^*)\delta I_6 \quad \textbf{... (58)}$$

We takes these variations one by one. Taking first the variation δI_1 and δI_2, integrating this and using the boundary condition.

$$I_1 = \int_0^1 \delta[|DW|^2 + a^2|W|^2]dz$$

or $$\delta I_1 = -\int_0^1 [(D^2 - a^2)\delta W^* dz - \int_0^1 (D^2 - a^2)\delta W]W^* dz. \quad \textbf{...(59)}$$

$$\delta I_3 = \int_0^1 \delta |M|^2 dz$$

$$= \int_0^1 M\delta M^* dz + \int_0^1 M^* \delta M dz$$

$$= \int_0^1 M[(D^2 - a^2)]\delta h_z^* dz + \int_0^1 M^*[D^2 - a^2)]\delta h_z dz$$

or $$\delta I_3 = \int_0^1 [(D^2 - a^2)^2 h_z]\delta h_z^* dz + \int_0^1 [(D^2 - a^2)^2 \delta h_z]\delta h_z^* dz. \quad \textbf{... (60)}$$

$$\delta I_4 = \int_0^1 [|DH_z|^2 + a^2 |h_z|^2 dz$$

or $$\delta I_4 = \int_0^1 [D^2 - a^2)\, h_z]\delta h_z^* dz - \int_0^1 [(D^2 - a^2)\delta h_z]\delta h_z^* dz. \quad \textbf{... (61)}$$

$$\delta I_5 = \int_0^1 |D\theta|^2 dz$$

or $$\delta I_5 = -\int_0^1 D^2 \delta\theta^* \theta dz - \int_0^1 D^2 \theta^* \delta\theta dz \quad \textbf{... (62)}$$

and $$\delta I_6 = \int_0^1 \delta |\theta|^2 d_z = \int_0^1 D^2 \delta\theta^* \theta dz + \int_0^1 D^2 \theta^* \delta\theta dz\,. \quad \textbf{... (63)}$$

Substituting the values of variations form equations **(59)** to **(63)** into the equation **(57)** and **(58)**, we get

$$2\delta I = \int_0^1 [-[P + R_d(FP - 1)](D^2 - a^2)W]\delta W^* dz$$

$$+Q\int_0^1 [(D^2 - a^2 - P_2 P^*)h_z^*](D^2 - a^2)h_z d_z$$

$$+\int_0^1 P + R_d(PF - 1)(D^2 - a^2)W^*]Wdz \; + Q\int_0^1 [D^2 - a^2 - P_2 P)\delta h_z](D^2 - a^2)h_z^* dz$$

$$+\int_0^1 -\{P + R_d(PF - 1)(D^2 - a^2)\delta W]W^* dz \; + Q\int_0^1 [D^2 - a^2 - P_2 P)h_z](D^2 - a^2)\delta h_z dz$$

$$+\int_0^1 -\{P + R_d(PF - 1)(D^2 - a^2)\delta W^*]Wdz \; + Q\int_0^1 [D^2 - a^2 - P_2 P)h_z](D^2 - a^2)\delta h_z^* dz$$

... (64)

and $$2\delta j = -\int_0^1 (D^2 - a^2 - P_1 O)\theta\delta\theta^* dz - \int_0^1 (D^2 - a^2 - P_1 P)\delta\theta\theta^* dz$$

$$-\int_0^1 (D^2 - a^2 - P_1 P^*)\theta^* \delta\theta\, dz - \int_0^1 (D^2 - a^2 - P_1 P^*)\delta\theta^* \theta dz. \quad \textbf{...(65)}$$

Substituting the values of δI and δJ from the equations **(64)** and **(65)** into the equation **(56),** we get

$$\delta R = \frac{1}{a^2 J}[\delta A_1 + \delta A_1^* + \delta A_2 + \delta A_2^*], \quad \textbf{...(66)}$$

Where $$\delta A_1. = \int \left[-\left[P + R_d\left(FP-1\right)\right](D^2 - a^2)W\right]\delta W^* dz$$
$$+Q\int_0^1 [(D^2 - a^2 - P_2 P^*)\delta h_z^*](D^2 - a^2)h_z dz$$
$$+a^2 R\int_0^1 (D^2 - a^2 - P_1 P^*)\theta\delta\theta^* dz \quad \textbf{...(67)}$$

and $$\delta A_2 = \int_0^1 \left[-\left[P + R_d(FP-1)\right](D^2 - a^2)\delta W\right]W^* dz$$
$$+Q\int_0^1 [(D^2 - a^2 - P_2 P^*)h_z^*](D^2 - a^2)\delta h_z dz$$
$$+a^2 R\int_0^1 (D^2 - a^2 - P_1 P^*)\theta^*\delta\theta dz. \quad \textbf{...(68)}$$

Let us first take up δA_1. For this taking the variation in δh_z and δW consistent equation **(39),** and taking its complex conjugate, we have

$$(D^2 - a^2 - P_2 P^*)\delta h_z^* = -D\delta W^*$$

Now, multiplying the above equation with $(D^2 - a^2)h_z$ and integrating, we have

$$\int_0^1 [D^2 - a^2 - P_2 P^*)\delta h_z^*](D^2 - a^2)h_z dz = -\int_0^1 D\delta W^*[D^2 - a^2)h_z]dz$$
$$= \int_0^1 [D(D^2 - a^2)h_z]\delta W^* dz. \quad \textbf{...(69)}$$

Again taking the variation is δq and δW in equation **(38)** and taking its complex conjugate, we have

$$(D^2 - a^2 - P_1 P^*)\delta\theta = -\delta W^*$$

Now multiplying the above equation with θ and integrating, we have

$$\int_0^1 [D^2 - a^2 - P_2 P^*)\delta\theta^*]\theta\, dz = -\int_0^1 \delta W^* \theta\, dz \quad \textbf{... (70)}$$

Using integrals **(69)** and **(70)** in δA_1 equation **(67),** we get.

$$\delta A_1 = \int_0^1 -\left[P + R_d\left(PF-1\right)\right](D^2 - a^2)W + \left\{QD(D^2 - a^2)h_z - a^2 RQ\right\}\delta W^* dz. \quad \textbf{...(71)}$$

Similarly, for δA_2 taking the complex conjugate of the equation **(39)**, we have

$$(D^2 - a^2 - P_2 P^*)h_z^* = -DW^*.$$

Multiplying this equation with $(D^2 - a^2)$ δh_z and integrating, we have

$$\int_0^1 [D^2 - a^2 - P_2 P^*) h_z^*](D^2 - a^2)\delta h_z dz = -\int_0^1 DW^*[D^2 - a^2]\delta h_z dz$$

$$= \int_0^1 [D(D^2 - a^2)\delta h_z] W^* dz. \quad \text{...(72)}$$

Using above integral into δA_2 equation **(68)** we get.

$$\delta A_2 = \int_0^1 \left[-\left\{ P + R_d \left(PF - 1 \right) \right\} (D^2 - a^2)\delta W + QD(D^2 - a^2)\delta h_z \right] W^* dz$$

$$+ a^2 R \int_0^1 (D^2 - a^2 - P_1 P^*)\theta^* \delta\theta dz. \quad \text{...(73)}$$

Taking the variation δW, δh_z and $\delta\theta$ in equation **(37)**, we have

$$\left[-\left\{ P + R_d \left(PF - 1 \right) \right\} (D^2 - a^2)\delta W + QD(D^2 - a^2)\delta h_z \right] = a^2 R \delta\theta. \quad \text{...(74)}$$

Using above variation **(74)** into the equation **(73),** we get

$$\delta A_2 = a^2 R \int_0^1 \left[W^* + (D^2 - a^2 - P_1 P^*)\theta^* \right] \delta\theta dz.$$

From the equation **(66),** we have

$$\delta R = \frac{1}{2a^2 J} \mathrm{Re}\left[\delta A_1 + \delta A_2 \right]$$

$$= \frac{1}{2a^2 J} \left[\int_0^1 -\left\{ P + R_d \left(PF - 1 \right) \right\} (D^2 - a^2) W + QD(D^2 - a^2) h_z - a^2 R \vartheta \right] \delta W^* dz$$

$$+ a^2 R \int_0^1 \left\{ W^* + (D^2 - a^2 - P_1 P^*)\theta^* \right\} \delta\theta dz.$$

Thus for the functions W, h_z and θ satisfying the equations **(37)** to **(39)** with boundary conditions, we see the $\delta R = 0$. This proves the variational principle.

SOLUTION OF THE PROBLEM FOR FREE BOUNDARIES

We consider the solution of the equations **(37)** to **(39)**, when the boundaries at $z = 0$ and 1 are free. We have established the variational principle in terms of Rayleigh number R and the solution of the given problem gives the extremum values of R over all possible functions satisfying the boundary conditions. We taken the function $W(z)$ as

$$W(z) = A \sin \pi z. \quad \text{...(75)}$$

Substituting this value of $W(z)$ from above equation **(75)** into equation **(38),** we have

$$(D^2 - a^2 - P_1 P)\theta = -A \sin \pi z \quad \text{... (76)}$$

Solving above equation **(76)** and using boundary conditions $\theta = 0$ at $z = 0$ and $z = 1$, we have

$$(D^2 - a^2 - P_1 P)\theta = -A \sin \pi z$$

$$\theta = \frac{A \sin \pi z}{(\pi^2 + a^2 + P_1 P)} = \frac{A \sin \pi z}{C_0} \quad \text{... (77)}$$

where $C_0 = (\pi^2 + a^2 + P_1 P)$

Again substituting the value of $W(z)$ from the equation **(75)** into the equation **(39)**, we have

$$(D^2 - a^2 - P_2 P) h_z = -A\pi \cos \pi z \quad \text{... (78)}$$

Solving the above equation **(78)** and using boundary conditions $Dh_z = 0$ at $z = 0$ and 1, we have

$$h_z = \frac{A\pi \cos \pi z}{\pi^2 + a^2 + P_2 P} = \frac{A\pi \cos \pi z}{D_0} \quad \text{... (79)}$$

where, $D_0 = \pi^2 + a^2 + P_2 P$.

Substituting the values $W(z)$, $\theta(z)$ and $h(z)$ from the equations **(75), (77)** and **(79)** into the equation **(37)**, we get

$$\left[\{P + R_d (PF - 1)\}\right] C_1 \sin \pi z + \frac{Q\pi^2 C_1}{D_0} \sin \pi z = \frac{a^2 R}{C_1 + P_1 P} \sin \pi z \quad \text{... (80)}$$

where $C_1 = \pi^2 + a^2$

Multiplying above equation **(80)** by sin πz and integrating from $z = 0$ to $z = 1$, we get the dispersion relation as

$$\left[\{P + R_d (PF - 1)\}\right] C_1 + \frac{Q\pi^2 C_1}{C_1 + P_2 P} = \frac{a^2 R}{C_1 + P_1 P} \quad \text{... (81)}$$

At the marginal state, $P_r = 0$ and so we express $P = Ip$. Substituting this in equation **(81),** we have

$$\left[\{iP + R_d (iPF - 1)\}\right] C_1 + \frac{Q\pi^2 C_1 (C_1 - iP_2 P)}{C_1^2 + P_2^2 P^2} = \frac{a^2 R (C_1 - iP_1 P)}{C_1^2 + P_1^2 P^2}. \quad \text{... (82)}$$

Separating the real and imaginary parts of the above equation **(82),** we have the real part as

$$-R_d C_1 + \frac{Q\pi^2 C_1}{C_1^2 + P_2^2 P^2} = \frac{a^2 R}{C_1^2 + P_1^2 P^2} \quad \text{...(83)}$$

And imaginary part

$$P_i\left[C_1 + R_d F C_1 + \frac{Q\pi^2 P_2 C_1}{C_1^2 + P_2^2 P^2} - \frac{a^2 R P_1}{C_1^2 + P_1^2 P^2}\right] = 0\,. \qquad \text{... (84)}$$

Eliminating the term between the equations **(83)** and **(84)**, we have

$$P_i\left[C_1 + \frac{Q\pi^2 C_1 (P_2 - P_1)}{C_1^2 + P_2^2 P^2} + C_1 R_d (F + P_1)\right] = 0\,. \qquad \text{...(85)}$$

If $P_i \neq 0$, then we have

$$\left[C_1 + \frac{Q\pi^2 C_1 (P_2 - P_1)}{C_1^2 + P_2^2 P^2} + C_1 R_d (F + P_1)\right] = 0\,.$$

From equation **(83),** we get the value of *R* as

$$R = \frac{C_1^2 + P_2^2 P^2}{a^2}\left[-R_d C_1 + \frac{Q\pi^2 C_1}{C_1^2 + P_2^2 P^2}\right]. \qquad \text{... (86)}$$

Now, we have fix the values of physical parameters Q, R_d, P_1, P_2 and then take the value of the wave number a and F, calculate the value of P from the equation **(85)** as its roots and then with these values of a and P, calculate R from **(86).** In this way we get the relation between R and a. The minimum value of R is the critical wave number a_c and wave frequency P_c for the fixed values of non-dimensional numbers.

CONCLUSION :

We have discussed the stability of the fluid layers confined between free boundaries. We find the sufficient condition for stability of the system is

$$R < \left[\frac{-R_d I_1 + Q I_3}{a^2 (I_5 + a^2 I_6)}, \frac{I_1 + R_d F I_1 + Q P_2 I_4}{a^2 P_1 I_4}\right]. \qquad \text{...(87)}$$

Fixing the values of parameters, Q, R_d, P_1, P_2, we determine the value of Rayleigh number R for frequency P and any value of the wave number a.

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