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THERMAL INSTABILITY OF A CONTINUOUSLY STRATIFIED VISCO-ELASTIC MAXWELL FLUID IN HYDROMAGNETICS THROUGH A POROUS MEDIUM

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Thermal instability of Rivlin-Ericksen elastico-viscous fluid in hydro magnetic field has been studied by various researchers. With the developing significance of non-Newtonian fluid in contemporary day-era and industries the investigations on continuously stratified visco-elastic fluid in presence of magnetic field in porous medium are desirable. In the present paper, thermal instability of continuously stratified visco-elastic Maxwell fluid in a porous medium in the presence of magnetic field is considered. Following linearized perturbation theory and normal mode technique, the dispersion relation is obtained. The system is found stable everywhere in flow domain if $R < 0$ and $(D\rho) < 0$. The non-oscillatory modes are unstable if $(D\rho) > 0$ everywhere in flow domain under the certain conditions. Oscillatory modes are stable if $R < 0$ and $(D\rho) > 0$ under certain conditions.

Keywords: Thermal instability, Stratified visco-elastic fluids, Maxwell fluid, Magnetic field, Porous medium.

INTRODUCTION

A detailed account of thermal instability in fluids and stability of superposed fluids under varying assumptions of hydrodynamic and hydromagnetics has been given by Chandrasekhar **[1]**. An authoritative to this fascinating subject has been discussed in detail in the celebrated monograph by Rosensweig **[2]**. This review several applications of heat transfer through ferromagnetic fluids. One such phenomenon is enhanced by convective cooling having a temperature dependent magnetic moment due to magnetization of the fluid. He showed magnetization in general, is the function of magnetic field, temperature and density of the fluid. Sharma and Kumar **[3]** have studied the thermal instability of Rivlin-Ericksen elastico-viscous fluid in hydromagnetics whereas the thermal convection in Rivlin-Ericksen visco-elastic fluid in porous medium in hydromagnetics has been studied by Sharma and Kango **[4]**. Sharma and Sharma **[5]** have studied the thermal instability in a Maxwellian visco-elastic fluid in porous medium. It is found that for stationary convection, Maxwellian fluid behaves like a Newtonian fluid and critical Rayleigh number increase with the increase in magnetic field and rotation.

The problem of thermaosolutal instability of a Oldroydian visco-elastic fluid in porous medium has been discussed by Sharma and Bhardwaj **[6]**. They found that stable solute gradient and rotation has a stabilizing effect on the system. Sharma and Kumar **[7]** have investigated the problem of thermal convection in Oldroydian visco-elastic fluid in porous medium.

The problem of thermosolutal instability of Rivlin-Ericksen visco-elastic fluid mixture in porous medium in the presence of magnetic field is discussed by Pundir **[8]**. Kumar and Lal **[9]** studied thermal instability of Walters' B visco-elastic fluid permeated with suspended particles under the effect of hydromagnetic field in porous medium. Kumar and Mohan **[10]** have investigated the effect of magnetic field on thermal instability of a rotating Rivlin-Ericksen visco-Elasitc fluid.

Now, it would be much interest to examine the thermal instability of continuously stratified visco-elastic Maxwell fluid in porous medium in presence of magnetic field. This topic seems to be uninvestigated so far. Various instability problems of such fluids have growing importance in modern technology and industry, geophysics and bio-mechanics.

CONSTITUTIVE EQUATIONS

The basic equations are,

$$\frac{\rho}{\varepsilon}\left[1+\lambda\frac{\partial}{\partial t}\right]\left[\frac{\partial \mathbf{q}}{\partial t}+\frac{1}{\varepsilon}(\mathbf{q}.\nabla)\,\mathbf{q}\right]=\left[1+\lambda\frac{\partial}{\partial t}\right]\left[-\nabla p+\rho X_i+\frac{\mu_e}{4\pi}(\nabla\times\mathbf{H})\times\mathbf{H}\right]-\frac{\mu}{k_1}\mathbf{q} \quad \text{... (1)}$$

$$\nabla.\mathbf{q}=0, \quad \text{... (2)}$$

$$\varepsilon\frac{\delta T}{\delta t}+(\mathbf{q}.\nabla)T=\kappa_T\nabla^2 T\,, \quad \text{... (3)}$$

$$\varepsilon\frac{\delta \mathbf{H}}{\delta t}=(\mathbf{H}.\nabla)\mathbf{q}+\varepsilon\eta\nabla^2\mathbf{H}\,, \quad \text{... (4)}$$

$$\varepsilon\frac{\delta\rho}{\delta t}+(\mathbf{q}.\nabla)\rho=0\,, \quad \text{... (5)}$$

$$\rho=\rho_0[1+\alpha(T_0-T)] \quad \text{... (6)}$$

and $$\nabla.\mathbf{H} = 0 \quad \textbf{... (7)}$$

where ε is the medium porosity, k_T is the thermal conductivity of the fluid, and $\beta = \frac{T_1 - T_0}{d}$ is the magnitude of uniform temperature gradient which is maintained and is positive as the temperature increases upward and T_0 are respectively the density and the temperature at the lower boundary $z = 0$.

BASIC STATE AND PERTURBATION EQUATIONS

The time independent solution of equation **(1)** to **(7)**, whose stability we wish to examine is that of an incompressible, electrically conducting, Maxwell visco-elastic fluid of varying density and variable viscosity arranged in horizontal strata in a homogeneous and isotropic porous medium. The system is acted upon by a uniform horizontal magnetic field **H** (H, 0, 0), a temperature T and gravity field **g** (0, 0, $-g$). The character of equilibrium is examined by supposing that the system is slightly disturbed and then by following its further evolution.

Let $\delta\rho, \delta p$, **q** (u, v, w), θ and **h** (h_x, h_y, h_z) denote respectively the perturbations in density ρ, pressure p, velocity (0, 0, 0), temperature T and the magnetic field **H** (H, 0, 0). Then the linearized perturbation equations are,

$$\frac{\rho}{\varepsilon}\left[1+\lambda\frac{\partial}{\partial t}\right]\frac{\partial u}{\partial t} = \left[1+\lambda\frac{\partial}{dt}\right]\left[-\frac{\partial}{\partial x}\delta p\right] - \frac{\mu}{k_1}u, \quad \text{... } \textbf{(8)}$$

$$\frac{\rho}{\varepsilon}\left[1+\lambda\frac{\partial}{\partial t}\right]\frac{\partial v}{\partial t} = \left[1+\lambda\frac{\partial}{dt}\right]\left[-\frac{\partial}{\partial y}\delta p + \frac{\mu_e H}{4\pi}\left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right)\right] - \frac{\mu}{k_1}v, \quad \text{... } \textbf{(9)}$$

$$\frac{\rho}{\varepsilon}\left[1+\lambda\frac{\partial}{\partial t}\right]\frac{\partial w}{\partial t} = \left[1+\lambda\frac{\partial}{dt}\right]\left[-\frac{\partial}{\partial y}\delta p + g\alpha\rho_0\theta - g\delta\rho + \frac{\mu_e H}{4\pi}\left(\frac{\partial h_z}{\partial x} - \frac{\partial h_x}{\partial z}\right)\right] - \frac{\mu}{k_1}w, \quad \text{... } \textbf{(10)}$$

$$\frac{\partial\theta}{\partial t} - \beta w = k_T\nabla^2\theta\,, \quad \text{... } \textbf{(11)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad \text{... } \textbf{(12)}$$

$$\varepsilon\frac{\partial}{\partial t}\partial\rho = -w(D\rho), \quad \text{... } \textbf{(13)}$$

$$\frac{\partial h_x}{\partial x}+\frac{\partial h_y}{\partial y}+\frac{\partial h_z}{\partial z}=0\,, \qquad \text{... (14)}$$

$$\frac{\partial h_x}{\partial t}=H\frac{\partial u}{\partial z}+\eta\nabla^2 h_x, \qquad \text{... (15)}$$

$$\frac{\partial h_y}{\partial t}=H\frac{\partial v}{\partial z}+\eta\nabla^2 h_y \qquad \text{... (16)}$$

and

$$\frac{\partial h_z}{\partial t}=H\frac{\partial w}{\partial z}+\eta\nabla^2 h_z. \qquad \text{... (17)}$$

Equation **(13)** results from the fact that the density of every fluid particle remains unchanged during its motion.

Analyzing the perturbations into normal modes, we seek solutions, whose dependence on *x*, *y* and *t* is given by

$$e^{\left\{i\left(k_X x+k_y y\right)+nt\right\}}, \qquad \text{... (18)}$$

Where k_x and k_y are horizontal wave numbers, $k=\sqrt{k_x^2+k_y^2}$ is the resultant wave number and *n* is complex in general.

With this dependence of perturbation, equations **(8)** to **(17)** reduce to

$$\frac{\rho}{\varepsilon}\left[1+\lambda n\right]nu=[1+\lambda n)[-ik_x\delta p]-\frac{u}{k_1}u, \qquad \text{... (19)}$$

$$\frac{\rho}{\varepsilon}[1+\lambda n)nv=[1+\lambda n]\left[-ik_y\delta p+\frac{u_e H}{4\pi}(ik_x h_y-ik_y h_x)\right]-\frac{u}{k_1}v\,, \qquad \text{... (20)}$$

$$\frac{\rho}{\varepsilon}[1+\lambda n)nw=[1+\lambda n]\left[-D\delta p+g\delta\rho++g\alpha\rho_0\theta+\frac{u_e H}{4\pi}(ik_x h_z-Dh_x)\right]-\frac{u}{nk_1}w, \qquad \text{... (21)}$$

$$n\theta-\beta w=k_T(D^2-k^2)\theta, \qquad \text{... (22)}$$

$$k_x u+k_y V=iDw, \qquad \text{... (23)}$$

$$k_x h_x+k_y h_y=iDH_z\,, \qquad \text{... (24)}$$

$$\varepsilon\, n\,\delta p=-w(D\rho), \qquad \text{...(25)}$$

$$nh_x=HDu+\eta(D^2-k^2)h_x, \qquad \text{... (26)}$$

$$nh_y=HDv+\eta(D^2-k^2)h_y \qquad \text{... (27)}$$

and $$nh_z = HDw + \mu(D^2 - k^2)h_z \,. \qquad \text{... (28)}$$

On adding equations **(19)** and **(20)** after multiplying by k_x and k_y respectively, we have,

$$\frac{n}{\varepsilon}[1+\lambda n]D(\rho Dw) = [1+\lambda n]\left[-k^2 D\delta p + i\frac{\mu_e H}{4\pi}k_y D\left(k_x h_y - k_y h_x\right)\right] - \frac{1}{k_1}[D(\mu Dw)] \qquad \text{...(29)}$$

Now subtracting equation **(21),** after multiplying it by k^2, from equation **(29),** we have

$$[1+\lambda n]D(\rho Dw) - k^2\rho w = [1+\lambda n]\left[\frac{-gk^2(D\rho)w}{n^2} - \frac{g\alpha k^2\varepsilon\rho_0\theta}{n} - \frac{\mu_e H^2 k_x^2(D^2-k^2)w}{4\pi n^2}\right]$$

$$-\frac{\varepsilon}{nk_1}\left[D(\mu Dw) - k^2\mu w\right]. \qquad \text{... (30)}$$

Further, equation **(22)** can be rewritten as

$$\left[n - k_T(D^2-k^2)\theta\right] = \beta w \,. \qquad \text{... (31)}$$

Now, using the non-dimensional quantities defined by

$$D^* = dD, \sigma = \frac{nd^2}{v}, a = kd \,.$$

And dropping the (*) for convenience, equations **(30)** and **(31)** become

$$(1+L\sigma)\left\{D(\rho.Dw) - a^2\rho w) + \frac{R_1 a^2(D\rho)w}{\sigma^2} + \frac{F\rho_0}{\sigma} + \frac{\rho_0 Q a_x^2}{\sigma^2}(D^2-a^2)w\right\}$$

$$+\frac{1}{p\sigma v}\left[D(\mu Dw) - a^2\mu w\right] = 0 \qquad \text{... (32)}$$

and $$(D^2 - a^2 - P_t\sigma)F = -Ra^2\varepsilon w \,. \qquad \text{...(33)}$$

where $R = \dfrac{g\alpha\beta d^4}{k_T v}$ is the thermal Rayleigh number,

$P_r = \dfrac{v}{k_T}$ is the thermal Prandtl number,

$Q = \dfrac{\mu_e H^2 d^2}{4\pi\rho_0 v^2}$ is the magnetic force number

and $$F = \frac{g\alpha a^2 d^2\theta}{v}, R_1 = \frac{gd^3}{v^2}, P = \frac{k_1}{\varepsilon d^2}, L = \frac{\lambda v}{d^2}, L_0 = \frac{\lambda_0 v}{d^2} \,.$$

Multiplying equation **(32)** by w*, integrating over the range of z and using equation **(33),** we have

$$(1+L\sigma)\left\{\left[1+\frac{Qa_x^2}{\sigma^2}\right]I_1-\frac{R_1a^2}{\sigma^2}I_2-\frac{\rho_0}{Ra^2\sigma}\left[I_3+(a^2+P_r)\sigma^* I_5\right]\right\}+\frac{1}{p\sigma v}I_4=0. \qquad \text{... (34)}$$

where

$$I_1=\int_0^1\rho(|Dw|^2+a^2|w|^2)dz,\ \ I_2=\int_0^1(D\rho)|w|^2\,dz,\ \ I_3=\int_0^1|DF|^2\,dz,$$

$$I_4=\int_0^1\mu\,(|Dw|^2+a^2|w|^2)dz \ \text{ and } \ I_5=\int_0^1|F|^2dz.$$

Integrals I_1, I_2, I_3, I_4 and I_5 are positive definite and I_2 is definitely positive or negative according as $D\rho$ is everywhere positive or everywhere negative. Further, equation **(34)** can also be written as

$$L\sigma^3I_1+\left[I_1\sigma^2-\frac{L\rho_0}{Ra^2}\left(I_3\sigma^2+(a^2+P_r)\,|\sigma^2\,|.\sigma I_5\right)\right]$$

$$+\sigma\left[LQa_x^2I_1-\frac{\rho_0}{Ra^2}I_3-\frac{\rho_0}{Ra^2}(a^2+P_r)L\,|\sigma|^2\,I_5+\frac{1}{Pv}I_4-LR_1a^2I_2\right]$$

$$+\left\{LQa^2I_1-R_1a^2I_2-\frac{\rho_0}{Ra^2}(a^2+P_r)|\sigma|^2\,I_5\right\}=0. \qquad \text{...(35)}$$

ANALYTICAL DISCUSSION

Depending upon various physical parameters, we obtain below a number of results stating clearly the role of these parameters.

Theorem-1. If R < 0 and ($D\,\rho$) < 0 everywhere in the flow domain, then the system is stable.

Proof: Observe that if R < 0 and ($D\,\rho$) < 0 everywhere in the flow domain, then equation **(35)** does not allow any positive value of σ . Neither it allows σ to be zero, so that σ can take only negative values, implying thereby that the system is stable.

This result being independent of magnetic field, hold even in the absence of magnetic field.

Theorem-2. If R < 0 and ($D\,\rho$) > 0 everywhere in the flow domain, then the system is stable. Under the condition $Qa_x{}^2I_1>R_1a^2I_2$.

Proof: Assuming that if $R < 0$ and $(D\rho) > 0$ everywhere in the flow domain, then the stability of the system is ensured under the condition $Qa_x{}^2 I_1 > R_1 a^2 I_2$

Theorem-2 ensures the stability of the system under the condition.

$$Q > R_1 \frac{a^2 I_2}{a_x^2 I_1} = Q^*$$

Thus given any unstable disturbance (hence any unstable wave number), a suitable magnetic force number Q can be obtained which will stabilize this disturbance. However, instability might occur when Q < Q*, though, we are unable to prove the instability in general when Q < Q*. Following two theorems are important in as much as they provide the instability of non-oscillatory modes and the number of stable and unstable modes under this condition.

DISCUSSION OF NON-OSCILLATORY MODES

Theorem -3: The non-oscillatory modes (if exist) are unstable when $(D\rho) > 0$ everywhere in the flow domain provided $Qa_x^2 < R_1 \frac{a^2 I_2}{I_1}$.

Proof: For non-oscillatory modes we have $\sigma_i = 0$. Then equation **(35)** reduces to

$$A\sigma_r^3 + B\sigma_r^2 + C\sigma_r - D = 0 \qquad \text{... (36)}$$

where $A = \left[LI_1 - \frac{L\rho_o}{Ra^2}(a^2 + P_r)I_5 \right]$, $B = \left[-\frac{L\rho_0}{Ra^2} I_3 + I_1 - \frac{\rho_0}{Ra^2}(a^2 - P_r)I_5 \right]$,

$C = \left[LQa_x^2 I_1 - \frac{\rho_0}{Ra^2} I_3 + \frac{1}{Pv} I_4 - LR_1 a^2 I_2 \right]$ and $D = \left[R_1 a^2 I_2 - Qa_x^2 I_1 \right]$.

Equation **(36)** is a cubic equation in σ_r and if $\sigma_{r1}, \sigma_{r2}, \sigma_{r3}$ are the roots of this equation, if follows that the product of the roots become positive under the conditions $(D\rho) > 0$ and $Qa_x^2 < R_1 \frac{a^2 I_2}{I_1}$. Therefore, either-all three roots are positive or else one root is positive and two roots are negative. In both the situations, system becomes unstable. It follows that the non-oscillatory modes if exist under the conditions of theorem, are unstable.

It is important to observe that the instability of non-oscillatory modes is independent of whether *R* is positive or negative. Theorem (4) below, however, holds only when $R < 0$.

Theorem–4: If $R < 0$, there are three waves propagating for a given wave number: two damped and one amplified, under the conditions $(D\rho) > 0$ and $Qa_x^2 < R_1 \frac{a^2 I_2}{I_1}$

Proof: Let the roots of the equation **(36)** be σ_{ri} = 1, 2, 3. Then using the theory of equations we get

$$\sigma_{r1}.\sigma_{r2}.\sigma_{r3} = \frac{D}{A} \text{ (Positive)}$$

and
$$\sigma_{r1} + \sigma_{r2} + \sigma_{r3} = -\frac{B}{A} \text{ (Negative)}$$

Clearly, when R < 0 and $Qa_x^2 < R_1 \frac{a^2 I_2}{I_1}$, then both A and B become positive definite so that the product of the roots is positive and the sum of the roots is negative. Therefore the possibility that all three non-oscillatory modes can be unstable is ruled out. It follows that two waves of propagation are damped and one is amplified for a given wave number.

Remark: In the absence of temperature effect (R = 0), equation **(35)** does not allow instability of non-oscillatory modes, If exist. Therefore, the temperature effects either induce instability of non-oscillatory modes or else there is no possibility of such modes to exist in the system. The possibility of the existence of non-oscillatory modes requires further investigations which are being undertaken.

Theorem 5: Unstable modes, if exist under the conditions R < 0 and $(D\rho)$ > 0 and $X > Y$ everywhere in the flow domain, are non-oscillatory.

where $X = \left[\frac{L\rho_0}{|R|a^2} I_3 + I_1\right]$ and $Y = \left[LQa_x^2 I_1 + \frac{\rho_0}{|R|a^2}(a^2 + P_r)I_4\right]$.

Proof: The imaginary part of equation **(35)**, taken after its division by σ, is given by

$$\sigma_i \left[2LI_1\sigma_r + I_1 + \frac{L\rho_o}{|R|a^2} I_3 + \frac{R_1 a^2 I_2}{|\sigma|^2} - LQa_x^2 I_1 - \frac{\rho_0}{|R|a^2}(a^2 + P_r)I_5 = 0\right]. \quad \text{...(37)}$$

Let the modes be unstable so that σ_r > 0, then for the consistency of equation **(37)** σ_i must be equal to zero, which ensure the existence of non-oscillatory modes. Hence unstable modes; if exist under the conditions R < 0, $X > Y$, and $(D\rho)$ > 0 everywhere in the flow domain, are non-oscillatory.

Theorem–6: Oscillatory modes, if exist under the conditions R < 0, $X > Y$ and $D\rho$ > 0 are stable, where X and Y are defined above.

Proof: For oscillatory modes $\sigma_i \neq 0$. Then for unstable modes σ_r < 0 so $2LI_1\sigma_r$ is negative therefore for consistency of equation **(37)** we have $X > Y$; R < 0 and $(D\rho)$ > 0.

Theorem 7: If the unstable modes exist under the conditions $R < 0$, $(D\rho) > 0$ and $X < Y$, then the bounds on σ_r for these unstable modes, are given by $\sigma_r < \frac{y}{2L}$.

Proof: For oscillatory modes $(\sigma_i \neq 0), \sigma_r > 0$ then $2L\sigma_r > 0$ therefore for the consistency of equations **(37)** we must have $2L\,\sigma_r - Y < 0$ or $\sigma_r < \frac{Y}{2L}$.

CONCLUSION :

Thermal instability of Rivlin-Ericksen elastico-viscous fluid in hydromagnetic field has been studied by Sharma and Kumar **[3]**. With the growing importance of non-Newtonian fluid in modern technology and industries the investigations on continuously stratified visco-elastic fluid in presence of magnetic field in porous medium are desirable. In the present paper, thermal instability of continuously stratified visco-elastic Maxwell fluid in a porous medium in the presence of magnetic field is considered. Following linearized perturbation theory and normal mode technique, the dispersion relation is obtained. The system is found stable everywhere in flow domain if $R < 0$ and $(D\rho) < 0$. The non oscillatory modes are unstable if $(D\rho) > 0$ everywhere in flow domain under the certain conditions. Oscillatory modes are stable if $R < 0$ and $(D\rho) > 0$ under certain conditions.

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