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DELIBERATION ON IDEAL TOPOLOGICAL GROUPS

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The ulterior motive of this article is to explore some properties of $\mathcal{I}$-interior and $\mathcal{I}$-closure via Ideal Topological Groups with good-enough counter examples. Also, we examine that under what condition $\mathcal{I}$-component of *e* is normal subgroup of *G*.

Keywords : Ideal topological space, Ideal topological group, $\mathcal{I}$-interior, $\mathcal{I}$-closure.

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INTRODUCTION

According to 1945 article of Montgomery in" The American Mathematical" monthly titled " What is a topological group?", the idea of topological group started with Sophus *L i.e.* The concept of Ideal topological spaces has been introduced and studied by Kuratowski and Vaidyanathaswamy. In 1990, Jankovic and Hamlett developed new topologies from old via ideals and introduced $\mathcal{I}$ - open sets with respect to an ideal in 1992. Combain the concepts Ideals and topological groups, S. Jafari and N. Rajesh introduce the new concept namely," Ideal Topological Groups"[1]. Here we concentrate on $\mathcal{I}$ - interior and $\mathcal{I}$ - closure.

PRELUDES

We bestow some prelude definitions with examples which are used to develop the theory of Ideal Topological Group.

Definition 2.1. Let (Y, τ) be the topological space and let $\mathcal{I}$ be the non-empty collection of subsets of *Y*. The set $\mathcal{I}$ is said to be an **Ideal** on the topological space (Y, τ) if it satisfies the following conditions. Are,

1. for $P \in \mathcal{I}$ and $Q \in P$ then $Q \in \mathcal{I}$.

2. for $P \in \mathcal{I}$ and $Q \in \mathcal{I}$ then $P \cup Q \in \mathcal{I}$.

If $\mathcal{I}$ is an ideal on Y then (Y,τ, $\mathcal{I}$)is called an ideal topological space.

Definition 2.2. Let $(Y, \tau, \mathcal{I})$ be an ideal topological space and $P(Y)$ be the collection of all subsets of Y. Now we define a function $^*(.) : P(Y) \longrightarrow P(Y)$by

$$^*P(\tau, \mathcal{I}) = \{y \in Y / U \cap P \notin \mathcal{I} \ \forall\ U \in \tau\,(y)\}, \text{where } \tau\,(y) = \{U \in \tau \,/\, y \in U\}.$$

This * (.)is called "**Local Function**" of P with respect to τ and $\mathcal{I}$. Instead of $^*P(\tau, \mathcal{I})$ we can use $^*P$.

Definition 2.3. For every ideal topological space there is a topology $^*\tau\,(\mathcal{I})$finer than τ, which is generated by $\delta(\mathcal{I}, \tau) = \{U - P \,/\, U \in \tau$ and $P \in \mathcal{I}\}$. But $\delta(\mathcal{I}, \tau)$ is not always a topology. A **Kuratowski Closure Operator** is defined as, $^*cl\ P = P \cup {}^*P\,(\mathcal{I}, \tau)$.

Example 2.4. Let $X = \{1, 2, 3, 4, 5\}$.

$\tau = \{\emptyset, X, \{1\}, \{2,5\}, \{1,2,5\},\{3\}, \{1,3\}, \{2,3,5\}, \{1,2,3,5\}\}$,

$\mathcal{I} = \{\emptyset,\{3,4,5\},\{3\}, \{4\}, \{5\}, \{3,4\}, \{4,5\}, \{3,5\}\}$

and $J = \{1, 3, 4, 5\}$.

Check whether $1 \in {}^*J$:

$\{1\} \cap J = \{1\} \notin \mathcal{I}$; $\{1,2,5\} \cap J = \{1,5\} \notin \mathcal{I}$; $\{1,3\} \cap J = \{1,3\} \notin \mathcal{I}$;

$\{1,2,3,5\} \cap J = \{1,3,5\} \notin \mathcal{I}$;$\{X\} \cap J = J \notin \mathcal{I}$.

For every $U \in \tau\,(1)$, $\underline{U} \cap J \notin \mathcal{I}$, therefore $1 \in {}^*J$.

Similarly, we can find all members of $^*J$, therefore $^*J = \{1,4\}$.

Kuratowski Closure, $^*cl\,(J) = J \cup {}^*J = \{1,3,4,5\} \cup \{1,4\} = \{1,3,4,5\}$.

Example 2.5. Let $X = \mathbb{R}$.

$$\tau = \{(a, t) \,/\, a, t \in \mathbb{R}\} \text{ and}$$

$$\mathcal{I} = \{\emptyset, \mathbb{Z}, P(\mathbb{Z})\}.$$

Now Let $J = \mathbb{Z}$.

Since for every $a \in X$, $U \cap J \in \mathcal{I}$ for some $U \in \tau(a)$, $^*J = \emptyset$.

The Kuratowski Closure, $^*cl\,(J) = J \cup {}^*J = \mathbb{Z} \cup \emptyset = \mathbb{Z}$.

Definition 2.6. Let P be a subset of an ideal topological space $(Y, \sigma, \mathcal{I})$. And let p be a point in P. If P □ int ($^*P$) then P is said to be ***$\mathcal{I}$* - open set**. The compliment of $\mathcal{I}$ - open set is called ***$\mathcal{I}$* - closed set**. If there exist $U \in \mathcal{I}_O(Y)$ such that $p \in U$ □ P then p is said to be an ***$\mathcal{I}$* - interior point** of P. The $\mathcal{I}$ - interior of P is defined as the collection of all $\mathcal{I}$ - interior points of P. If $U \cap P \neq \emptyset$ for every $U \in \mathcal{I}_O(Y)$ then p is said to be an **$\mathcal{I}$ -limit point** of P. The $\mathcal{I}$ -closure of P is defined as the collection of all $\mathcal{I}$ -limit point of P. If there exist $U \in \mathcal{I}_C(Y)$ such that $U \cap P = \{p\}$,then pis said to be an **$\mathcal{I}$ -isolated point** of P. Let N be the subset of an ideal topological space $(Y, \sigma, \mathcal{I})$. Then N is said to be **$\mathcal{I}$ - neighbourhood** of a point $y \in Y$ if there exist an $\mathcal{I}$ -open set Q such that $y \in Q$ □ N. If Nis an $\mathcal{I}$ - open set then N is said to be an **$\mathcal{I}$ - open neighbourhood** of a point y.

Example 2.7. Let $Y = \{1, 2, 3\}$

σ =Open sets of $Y = \{\emptyset, Y, \{1\}, \{2\}, \{1,2\}\}$; Closed sets of $Y = \{\emptyset, Y, \{2,3\},\{1,3\},\{3\}\}$ and $\mathcal{I} = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}$.

Let $P_1 = \emptyset$, obviously, P_1 is an $\mathcal{I}$ -open set.

Let $P_2 = \{1\}$, Then ${}^*P_2 = \{1,3\} \Rightarrow \text{int}({}^*P_2) = \{1\} \Rightarrow P_2$ □ $\text{int}({}^*P_2)$.

Let $P_3 = \{2\}$, Then ${}^*P_3 = \emptyset \Rightarrow \text{int}({}^*P_3) = \emptyset \Rightarrow P_3 \not\subset \text{int}({}^*P_3)$.

Let $P_4 = \{3\}$, Then ${}^*P_4 = \emptyset \Rightarrow \text{int}({}^*P_4) = \emptyset \Rightarrow P_4 \not\subset \text{int}({}^*P_4)$.

Let $P_5 = \{1,2\}$, Then ${}^*P_5 = \{1,3\} \Rightarrow \text{int}({}^*P_5) = \{1\} \Rightarrow P_5 \not\subset \text{int}({}^*P_5)$.

Let $P_6 = \{1,3\}$, Then ${}^*P_6 = \{1,3\} \Rightarrow \text{int}({}^*P_6) = \{1\} \Rightarrow P_6 \not\subset \text{int}({}^*P_6)$.

Let $P_7 = \{2,3\}$, Then ${}^*P_7 = \emptyset \Rightarrow \text{int}({}^*P_7) = \emptyset \Rightarrow P_7 \not\subset \text{int}({}^*P_7)$.

Let $P_8 = \{1,2,3\}$, Then ${}^*P_8 = \{1,3\} \Rightarrow \text{int}({}^*P_8) = \{1\} \Rightarrow P_8 \not\subset \text{int}({}^*P_8)$.

The collection of $\mathcal{I}$ -open sets of Yis, $\mathcal{I}_O(Y) = \{\emptyset, \{1\}\}$.

Also, the collection of $\mathcal{I}$ -closed sets of Y is, $\mathcal{I}_C(Y) = \{Y, \{2,3\}\}$.

Let $C = \{1,2\}$. Here $\mathcal{I}$ - int $C = \{1\}$ and $\mathcal{I}$ - cl $C = \{Y\}$.

Also 1is the $\mathcal{I}$ -interior point of C and 1, 2, 3 are the $\mathcal{I}$ - limit points of C.

Here 1 is the $\mathcal{I}$ - isolated point of C.

And {1, 2} is an $\mathcal{I}$ - neighbourhood of 1, but not an $\mathcal{I}$ - open neighbourhood of 1,

{1} is an $\mathcal{I}$ - open neighbourhood of 1.

Definition 2.8. Let $(Y, \sigma, \mathcal{I}_1)$ and $(Z, \tau, \mathcal{I}_2)$ be two ideal topological spaces and let

$h: (Y, \sigma, \mathcal{I}_1) \longrightarrow (Z, \tau, \mathcal{I}_2)$. The function his called to be

(i) **$\mathcal{I}$ - continuous** if $h^{-1}(U) \in \mathcal{I}_O(Y)$ for all $U \in \tau$.

(ii) **$\mathcal{I}$ - open** if $h(V) \in \mathcal{I}_O(Z)$ for all $V \in \mathcal{I}_O(Y)$.

(iii) **$\mathcal{I}$- closed** if $h(V) \in \mathcal{I}_C(Z)$ for all $V \in \mathcal{I}_C(Y)$.

Also, the function his called to be $\mathcal{I}$ - continuous $\Leftrightarrow$ for each $y \in Y$ and for every open neighbourhood U of $h(y)$, there is an $\mathcal{I}$ - open neighbourhood V of x such that

$h(V) \subset U$.

Example 2.9. Let $Y = \{l, m, n, o\}$

$$\sigma = \{\emptyset, Y, \{l\}, \{m\}, \{l,m\}, \{m,n\}, \{l,m,n\}, \{m,o\}, \{m,n,o\}, \{l,m,o\}\}.$$

$$\mathcal{I}_1 = \{\emptyset, \{n\}, \{o\}, \{n,o\}\}.$$

Here, $\mathcal{I}_O(Y) = \{\emptyset, Y, \{l\}, \{m\}, \{l,m\}, \{m,n\}, \{m,o\}, \{m,n,o\}, \{l,m,n\}, \{l,m,o\}\}$

And let $Z = \{1,2,3,4\}$.

$$\tau = \{\emptyset, Z, \{1\}, \{1,2\}, \{1,4\}, \{1,2,4\}\} \text{ and } \mathcal{I}_2 = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}.$$

Define $h: (Y, \sigma, \mathcal{I}_1) \longrightarrow (Z, \tau, \mathcal{I}_2)$ by $h(l) = 3$; $h(m) = 1$; $h(n) = 2$; $h(o) = 4$

Here $h^{-1}(\tau) = \{\emptyset, \{m\}, \{m,n\}, \{m,o\}, \{m,n,o\}, \{l,m,n\}\}$.

Thus, h is $\mathcal{I}$ - continuous.

Example 2.10. Let $Y = \{1, 2, 3\}$

$$\sigma = \{\emptyset, Y, \{1\}, \{2\}, \{1,2\}\}.$$

$$\mathcal{I}_1 = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}.$$

Here $\mathcal{I}_O(Y) = \{\emptyset, \{1\}\}$ and $\mathcal{I}_C(Y) = \{Y, \{2,3\}\}$.

And now let $Z = \{a,b,c,d\}$.

$$\tau = \{\emptyset, Z, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{b,d\}, \{b,c,d\}, \{a,b,d\}\}.$$

$$\mathcal{I}_2 = \{\emptyset, \{a\}, \{a, d\}, \{a,c\}, \{c\}, \{d\}, \{c,d\}, \{a,c,d\}\}.$$

Here $\mathcal{I}_O(Z) = \{\emptyset, \{b\}, \{b,c\}, \{b,d\}, \{b,c,d\}\}$.

and $\mathcal{I}_C(Z) = \{Z, \{a,c,d\}, \{a,d\}, \{a,c\}, \{a\}\}$.

Now we define $\phi: (Y, \sigma, \mathcal{I}_1) \rightarrow (Z, \tau, \mathcal{I}_2)$ by $\phi(1) = b$; $\phi(2) = a$; $\phi(3) = d$.

Here $\phi(\mathcal{I}_O(Y)) = \{\emptyset, \{b\}\}$. Thus, ϕ is $\mathcal{I}$ -open.

Now we define $g: (Y, \sigma, \mathcal{I}_1) \rightarrow (Z, \tau, \mathcal{I}_2)$ by $g(1) = c$; $g(2) = a$; $g(3) = d$.

Here $g(\mathcal{I}_C(Y)) = \{\{a,d\}, \{a,c,d\}\}$. Thus, g is $\mathcal{I}$ -closed.

Definition 2.11. Let $(Y, \sigma, \mathcal{I}_1)$ and $(Z, \tau, \mathcal{I}_2)$ be two ideal topological spaces and let $h : (Y, \sigma, \mathcal{I}_1) \longrightarrow (Z, \tau, \mathcal{I}_2)$. The function h is called $\mathcal{I}$ -homeomorphism if it is bijective, $\mathcal{I}$ -continuous and $\mathcal{I}$ -open.

RELATION BETWEEN OPEN SETS AND $\mathcal{I}$ - OPEN SETS

Here we are going to see, " Is there any relation between Open sets and $\mathcal{I}$ - Open sets?".

Result 2.12. The collection of open sets of an ideal topological space is need not to be the sub-collection of $\mathcal{I}$ -open sets of that ideal topological space.

Example 2.13. Let $Y = \{1, 2, 3\}$; σ = Open sets of $Y = \{\emptyset, Y, \{1\}, \{2\}, \{1,2\}\}$ and $\mathcal{I} = \{\emptyset, \{2\}, \{3\}, \{2,3\}\}$.

The collection of $\mathcal{I}$ -open sets of Y is, $\mathcal{I}_O(Y) = \{\emptyset, \{1\}\}$.

Here $\{2\} \in \sigma$ but not in $\mathcal{I}_O(Y)$.

Result 2.14. The collection of $\mathcal{I}$ -open sets of an ideal topological space is need not to the sub-collection of the open sets of that ideal topological space.

Example 2.15. Let $Y = \{1, 2, 3\}$; σ = Open sets of $Y = \{\emptyset, Y\}$ and $\mathcal{I} = \{\emptyset, \{1\}\}$.

The collection of $\mathcal{I}$ -open sets of Y is, $\mathcal{I}_O(Y) = \{\emptyset, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, Y\}$.

Here $\{2\} \in \mathcal{I}_O(Y)$ but not in σ.

Theorem 2.16. Let $(G, \tau, \mathcal{I})$ be an ideal topological space.

(i) If P be an open set with $P \subseteq {}^*P$, then Pis also an $\mathcal{I}$ -open set of G.

(ii) If P be an $\mathcal{I}$ -open set with ${}^*P \subseteq P$, then P is also an open set of G.

Proof: (i)- Let P be an open set and $P \subseteq {}^*P$.

$\Rightarrow$ $P = \text{int}(P) \subseteq \text{int}({}^*P)$, since P is open.

$\Rightarrow$ P is an $\mathcal{I}$ -open set

(ii) Let P be an $\mathcal{I}$ -open set and ${}^*P \subseteq P$.

$\Rightarrow$ $P \subseteq \text{int}({}^*P) \subseteq \text{int}(P)$, since $P \subseteq \text{int}({}^*P)$.

But always $\text{int}(P) \subseteq P$, $\Rightarrow P = \text{int}(P)$.

$\Rightarrow$ P is an open set

IDEAL TOPOLOGICAL GROUPS

We confront the definition of Ideal Topological Group and probe what are the trivial $\mathcal{I}$ -homeomorphisms, in this section.

Definition 3.1. A topological group is said to be an Ideal Topological Group if for each *a*, *b* ∈ *G* and each neighbourhood *C* of $a*b^{-1}$ there exist ℐ - open neighbourhood *A* of *a* and *B* of bsuch that $A * B^{-1}$ □ *C*.

Example 3.2. Let *Y* = {1, -1, *i*, -*i*} and τ ={∅, *Y*}.

Clearly, (*Y*,., τ) is a topological group, since a group with discrete or in-discrete topology is a topological group.

Now let ℐ = {∅, {*i*}, {-*i*}, {*i*, -*i*}}.

Here $\mathcal{I}_O(Y)$ = {∅, {1}, {-1}, {1,-1}, {1,*i*}, {1,-*i*}, {-1,*i*}, {-1, -*i*},

{1,-1,*i*}, {1,-1,-*i*}, {1,*i*, -*i*}, {-1,*i*,-*i*}, *Y*}.

Clearly (*Y*,., τ, ℐ) is an ideal topological group.

Example 3.3. Let *X* = {1, □, $□^2$}; τ = {∅, *X*, {1 }, { □ }, { $□^2$ }, {1, □ }, {1, $□^2$ }, { □, $□^2$ } } and ℐ = {∅}.

Here $\mathcal{I}_O(X)$ = {∅, *X*, {1}, {□ }, {$□^2$}, {1,□ }, {1, $□^2$}, {□, $□^2$}}.

Clearly (*X*,.,τ, ℐ) is an ideal topological group.

Theorem 3.4. Let (*G*, *, τ, ℐ)be an ideal topological group. Then the following are holds,

(i) $P \in \mathcal{I}_O(G) \Leftrightarrow P^{-1} \in \mathcal{I}_O(G)$.

(ii) If $P \in \mathcal{I}_O(G)$ and *Q* □*G*, then *P* * *Q* and *Q* * *P* are in $\mathcal{I}_O(G)$.

Proof. Let (*G*, *, τ, ℐ) be an ideal topological space.

(i) Let $P \in \mathcal{I}_O(G) \Leftrightarrow P$ □ int ({*p* ∈*G* / *U* ∩ *P* ∉ ℐ ∀ *U* ∈ τ (*p*)}). Now applying inverse map,ϕ on both sides, ⇔ϕ (*P*) □ ϕ(int ({*p* ∈ *G* / *U* ∩ *P* ∉ ℐ ∀ *U* ∈ τ (*p*)})). Since the inverse mapping is a homeomorphism,⇔ ϕ(*P*) □ int (ϕ({*p* ∈*G* / *U* ∩ *P* ∉ ℐ ∀ *U* ∈ τ(*p*)}))⇔P^{-1}□ int($^*(P^{-1})$)⇔ $P^{-1} \in \mathcal{I}_O(G)$.

(ii) Let <u>*P*</u> $\in \mathcal{I}_O(G)$ and *Q* □ *P*. By assumption we have, *P* □ int ($^*P$).

⇒ *P* □ int ({*p* ∈ *G* / *U* ∩ *P* ∉ ℐ ∀ *U* ∈ τ (p)}).

⇒ *P* * *q* □ int ({*p** *q* ∈*G* / *U**q* ∩ *P* ∉ ℐ ∀ *U* ∈ τ (*p*)}).

⇒ *P* * *Q* □ int ({*p***q* ∈ *G*/ (*U***V*) ∩ P∉ ℐ ∀ *U* * *V* ∈τ (*p* * *q*) where *q* ∈ *Q*}).

⇒ *P* * *Q* □ int ($^*(P * Q)$)⇒ $P * Q \in \mathcal{I}_O(G)$. Similarly, we can prove $Q * P \in \mathcal{I}_O(G)$.

Result 3.5. Let (*G*, *, τ, ℐ) be an ideal topological group. Then the left translation L_G and the right translation R_G are ℐ -homeomorphisms and the inverse map defined as ϕ(*g*) = g^{-1} for all *g* ∈*G*, is also an ℐ -homeomorphism.

PROPERTIES OF $\mathcal{I}$ - INTERIOR AND $\mathcal{I}$ - CLOSURE

We sketch out some properties of $\mathcal{I}$ -interior and $\mathcal{I}$ -closure on Ideal Topological Group, in this section.

Theorem 4.1. Let (G, *, τ, $\mathcal{I}$) be an ideal topological group and let P □ G Then

(i) $(\mathcal{I}\text{ int }(P))^{-1} = \mathcal{I}\text{ int }(P^{-1})$.

(ii) $(\mathcal{I}\text{ cl }(P))^{-1} = \mathcal{I}\text{ cl }(P^{-1})$.

Proof. (i)-RHS = $\mathcal{I}$ int (P^{-1})=$\mathcal{I}$ int (ϕ(P)), ϕ is the inverse mapping on G.

Since ϕ is $\mathcal{I}$ -homeomorphism, = $\phi(\mathcal{I}\text{ int }(P))= (\mathcal{I}\text{ int }(P))^{-1}$= LHS.

(ii)-RHS = $\mathcal{I}$ cl (P^{-1})= $\mathcal{I}$ cl (ϕ(P)), ϕ is the inverse mapping on G.

Since ϕ is $\mathcal{I}$ -homeomorphism, = ϕ ($\mathcal{I}$ cl(P))= $(\mathcal{I}\text{ cl }(P))^{-1}$= LHS.

Theorem 4.2. Let (G, *, τ, $\mathcal{I}$)be an ideal topological group and let P, Q □G. Then,

(i) $\mathcal{I}$ cl (P) * $\mathcal{I}$ cl (Q) □ cl (P*Q).

(ii) $\mathcal{I}$ int (P) * $\mathcal{I}$ int (Q) □ int (P*Q).

(iii) $(\mathcal{I}\text{ cl }(P))^{-1}$□ cl ($P^{-1}$).

(iv) $(\mathcal{I}\text{ int }(P))^{-1}$□ int ($P^{-1}$).

Proof. (i) Let $a * b \in \mathcal{I}$ cl (P) * $\mathcal{I}$ cl (Q)$\Rightarrow a \in \mathcal{I}$ cl (P) and $b \in \mathcal{I}$ cl (Q). Let C be the neighbourhood of $a*b$.Then there exist $\mathcal{I}$ -open neighbourhoods A of a and B of b such that $A * B$ □ C. Therefore,we have $x \in P \cap A$ and $y \in Q \cap B$, other than aand brespectively. Now we have, $x * y \in (P \cap A) * (Q \cap B) \in (P * Q) \cap C \Rightarrow x * y \in (P \cap C) * (Q \cap C) \Rightarrow a * b \in$ cl ($P * Q$) $\Rightarrow \mathcal{I}$ cl (P) * $\mathcal{I}$ cl (Q) □ cl (P*Q).

(ii) Let $a * b \in \mathcal{I}$ int (P) * $\mathcal{I}$ int (Q)$\Rightarrow$ $a \in \mathcal{I}$ int (P) and $b \in \mathcal{I}$ int (Q). Let C be the neighbourhood of $a*b$.Then there exist $\mathcal{I}$ -open neighbourhoods A of a and B of b such that A * B □ C . $\therefore$ we have, A □ P and B □ $Q \Rightarrow a * b \in A * B$ □ $P * Q \Rightarrow a * b$ is an interior point of $P * Q \Rightarrow a * b \in$ int ($P * Q$) $\Rightarrow \mathcal{I}$ int (P) * $\mathcal{I}$ int (Q) □ int (P*Q).

(iii) Let $a \in (\mathcal{I}\text{ cl }(P))^{-1}$and let Abe a neighbourhood of a. Since the inverse mapping is $\mathcal{I}$ -homeomorphism, there exist a set A^{-1}which is an $\mathcal{I}$ -open neighbourhood of a^{-1}.Since $a^{-1}\in\mathcal{I}$ cl (P), we have, $P \cap A^{-1} \neq \emptyset$. $\therefore$ $P^{-1} \cap A \neq \emptyset$, since the inverse map is one- one.$\Rightarrow$a is the limit point of $P^{-1}\Rightarrow(\mathcal{I}\text{ cl }(P))^{-1}$□ cl($P^{-1}$).

(iv) Let $a \in (\mathcal{I}\text{ int }(P))^{-1}$and let Abe a neighbourhood of a. Since $a^{-1} \in \mathcal{I}$ int (P), we have, A^{-1}□ P. Applying inverse mapping, A □ $P^{-1} \Rightarrow a$ is the interior point of $P^{-1} \Rightarrow (\mathcal{I}\text{ int }(P))^{-1}$ □ int (P^{-1}).

CONNECTEDNESS

We acquaint the definition of $\mathcal{I}$ -connectedness and some theorems on it, in this section.

Definition 5.1. Let (Y, τ, $\mathcal{I}$) be an ideal topological space. A set P □ Y is said to be **$\mathcal{I}$ -separated** if there exist two non-empty $\mathcal{I}$ -open sets M, N such that $M \cap N = \emptyset$ and $M \cup N = P$ or such that $\mathcal{I}$ cl (M) $\cap$ N = $\mathcal{I}$ cl (N) $\cap$ $M = \emptyset$ and $M \cup N = P$. A set P □ Y is said to be **$\mathcal{I}$ -connected** if it is not $\mathcal{I}$ -separated. Also, the space Y is said to be $\mathcal{I}$ -connected if there is no $\mathcal{I}$ -separation for Y.

Definition 5.2.A set P □ Y is said to be **$\mathcal{I}$ -component of an element 's'** if P is the union of all $\mathcal{I}$ -connected subsets of Y, each contain the 's'. That is P is the maximum $\mathcal{I}$ -connected subsets of Y, contains 's'. A set P □ Y is said to be **$\mathcal{I}$ -component of G** if P is the union of all $\mathcal{I}$ -connected subsets of Y. That is P is the maximum $\mathcal{I}$ -connected subsets of Y.

Example 5.3. Let $Y = \{a, b, c, d\}$

$$\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{b,d\}, \{b,c,d\}, \{a,b,d\}\}.$$

$$\mathcal{I}_1 = \{\emptyset, \{c\}, \{d\},\{c,d\}\}.$$

Here, $\mathcal{I}_O(Y) = \{\emptyset, Y, \{a\}, \{b\}, \{a,b\}, \{b,c\}, \{b,d\}, \{b,c,d\}, \{a,b,c\}, \{a,b,d\}\}$

and $\mathcal{I}_C(Y) = \{Y, \{b,c,d\}, \{a,c,d\}, \{c,d\}, \{a,d\}, \{a,c\}, \{a\}, \{d\}, \{c\},\emptyset\}$.

Now let $M = \{a,c\}$ and $N = \{d\}$.

$\Rightarrow$ $\mathcal{I}$ cl M = {a,c} and $\mathcal{I}$ cl N = {a,d}.$\Rightarrow \mathcal{I}$ cl $M \cap N = \emptyset$ and $\mathcal{I}$cl $N \cap M = \{a\} \neq \emptyset$.

∴ the pair M and Nis not $\mathcal{I}$ -separated.

∴ M ∪ N = $\{a,c,d\}$ □ Y is $\mathcal{I}$ - connected subset of Y.

Lemma5.4. The $\mathcal{I}$ -continuous image of $\mathcal{I}$ -connected space is $\mathcal{I}$ -connected.

Lemma5.5. Let $\{P_i\}$be the collection of $\mathcal{I}$ -connected subsets of an ideal topological space G. If $\bigcap_i P_i \neq \emptyset$ then $\bigcup_i P_i$ is $\mathcal{I}$ -connected.

Lemma5.6. If P is $\mathcal{I}$ -connected subset of an ideal topological space G. Let a subset Q of Gas P □ Q □$\mathcal{I}$ -cl (P). Then Q is also $\mathcal{I}$ - connected.

Lemma5.7. If P is $\mathcal{I}$ -connected then $\mathcal{I}$ -cl P is also $\mathcal{I}$ - connected.

Theorem5.8. Let (G, *, τ, $\mathcal{I}$) be an ideal topological group. Then every $\mathcal{I}$ - component is an $\mathcal{I}$ - closed set in G.

Proof. Let (G, *, τ, ℐ) be an ideal topological group and let P be an ℐ -component of G. By the previous lemma 5.7, ℐ -cl P is ℐ -connected. By the definition of ℐ - component,

P = ℐ -cl $P \Rightarrow P$ is ℐ - closed.

Theorem 5.9. Let (G, *, τ, ℐ) be an ideal topological group and let an ℐ -component of e in G and let all the right and left translations be ℐ -continuous. Then that ℐ -component of e is normal subgroup of G.

Proof. Let (G, *, τ, ℐ) be an ideal topological group and let P an ℐ - component of e. And let all the right and left translations be ℐ -continuous. First, we have to show, P is subgroup. It enough to prove that $P * P^{-1}$□ P. Since all translations are ℐ -continuous and by the lemma 5.4, all $P * y^{-1}$ is also ℐ -connected. Now we can write, $P * P^{-1} = \bigcup_{y \in P} \; P * y^{-1}$. Since $y \in P$, each $P * y^{-1}$ has e. That is, $\cap P * y^{-1} \neq \emptyset$, by the theorem 5.5, $\bigcup_{y \in P} \; P * y^{-1}$ is ℐ -connected. ⇒ $P * P^{-1}$is ℐ -connected and also contains e. But P is ℐ -component of e.

∴ P is the largest ℐ - connected subset of G that contains e. ⇒$P * P^{-1}$□ P. ⇒ P is subgroup. Now we have to show P is normal. That is to prove $G * P * G^{-1}$ □ P. Since all translations are ℐ -continuous and by the lemma 5.4, all $P * y^{-1}$ is also ℐ -connected. Now we can write, $G * P * G^{-1} = \bigcup_{g \in G} \; g * P * g^{-1}$. Also $\cap g * P * g^{-1} \neq \emptyset$. Then $\bigcup_{g \in G} \; g * P * g^{-1}$ is ℐ - connected.⇒G * P * G^{-1} is ℐ -connected and also contains e. ButPis ℐ -component of e. ∴ P is the largest ℐ -connected subset of G that contains e. ⇒$G * P * G^{-1}$□ P. ⇒ P is normal. There for ewe can conclude that P is normal subgroup.

Corollary 5.10. By the previous two theorems (5.8 and 5.9), we can conclude that P is ℐ -closed normal subgroup of G,

CONCLUSION :

In this paper, we analyses some properties of ℐ - interior and ℐ - closure on Ideal Topological Groups and give own examples to each definitions. Also, we examine that under what condition ℐ - component of is normal subgroup of G.

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