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INFINITESIMAL TRANSFORMATION IN A FINSLER SPACE

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This paper has been divided into three sections, of which the first section is introductory, and the second section deals with normal projective infinitesimal transformation and curvature collineation in a Finsler space. In this section we give the following definitions: affine motion, normal projective curvature collineation, Ricci normal projective curvature collineation and infinitesimal normal projective transformation, also we have derived results in the form of Lie derivatives of normal projective curvature tensor N^i_{jkh} and of the Ricci tensor N_{kh} and in this continuation we have derived certain more results the projective deviation tensor and its Lie derivatives. After these observations we have derived result in the form of theorems telling as to what will happen to the covariant vector fields $b_j(x,\dot{x})$ and $d_r(x,\dot{x})$ when the infinitesimal normal projective transformation $\bar{x}^i = x^i + v^i\ (x)dt$ defines an affine as well as a non-affine motion and also derived the results which will hold when the infinitesimal normal projective point transformation $\bar{x}^i = x^i + v^i\ (x)dt$ defines a normal projective curvature and normal projective Ricci collineations. In this continuation, we have also derived results which will hold good if the infinitesimal normal projective transformation $\bar{x}^i = x^i + v^i\ (x)dt$ itself is affine and non-affine and also we have derived if the Finsler space under consideration is symmetric. The third and the last section we have study of infinitesimal projective transformation with special reference to Cartan's connection $\Gamma^{*i}_{jk}(x,\dot{x})$ here the previous section we have taken in the form of normal projective connection $\Pi^i_{jk}(x,\dot{x})$

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the two connection coefficients are quite different so results will be different. After these observations we have derived results in the form of theorems if the infinitesimal transformation $\bar{x}^i = x^i + v^i\ (x)dt$ defines an affine motion then in such a case the vector fields $b_j(x, \dot{x})$ and $d_l(x, \dot{x})$ should separately vanish. In this continuation we have also derived the relationships which will hold when the infinitesimal transformation $\bar{x}^i = x^i + v^i\ (x)dt$ defines Cartan's curvature collineation as well as Cartan's Ricci collineation. In the last we have derived the relationships which will hold when the infinitesimal transformation $= x^i + v^i\ (x)dt$ is non-affine and affine one in a symmetric Finsler space.

Keywords: Normal projective infinitesimal transformation, Normal projective curvature collineation, Ricci normal projective curvature collineation, Lie-derivative, Cartan's curvature collineation, Cartan's Ricci collineation, Affine and non- affine motion, Symmetric Finsler space.

INTRODUCTION

Takano [9] who discussed the projective motion in a Riemannian space with bi-recurrent curvature and Yano and Nagano [12] defined projective conformal transformation in a Riemannian space and studies of curvature collineation and its properties in a Riemannian space have been carried out by Katzin, Levine and Davis [1].

Sinha [7] who defined the infinitesimal projective transformation in a Finsler space and by Pande and Kumar [4] who also defined infinitesimal transformation $\bar{x}^i = x^i + v^i\ (x)dt$ and derived some results in the form of theorems. An attempt to extend the theory of curvature collineation in Finsler space has been made by Singh and Prasad [8] and Pande and Kumar [5]. The relations which exist in a Finsler space admitted by curvature collineation and other symmetries have been studied by these researchers.

NORMAL PROJECTIVE INFINITESIMAL TRANSFORMATION AND CURVATURE COLLINEATION

Yano [10] has defined the Lie derivatives of any tensor field $T_j^i(x, \dot{x})$ and the connection coefficient $\Pi_{jk}^i(x, \dot{x})$ with respect to the normal projective covariant derivative respectively given as:

$$\pounds_v T_j^i = (\nabla_s T_j^i)v^s - T_j^s(\nabla_s v^i) + T_s^i(\nabla_j v^s) + (\dot{\partial}_s T_j^i)(\nabla_m v^s)\dot{x}^m \quad \text{... (2.1)}$$

and $$\pounds_v \Pi_{jk}^i = \nabla_j \nabla_k v^i + (\dot{\partial}_h \Pi_{jk}^i)(\nabla_s v^h)\dot{x}^s . \quad \text{... (2.2)}$$

Also we shall use the following commutation formulae in the form of the Lie-derivative and normal projective covariant derivative

$$\dot{\partial}_r(\pounds_v T_{jh}^i) - \pounds_v(\dot{\partial}_r T_{jk}^i) = 0 , \quad \text{... (2.3)}$$

$$\pounds_v(\nabla_r T_{jk}^i) - \nabla_r(\pounds_v T_{jk}^i) = T_{jk}^s \pounds_v \Pi_{sr}^i - T_{sk}^i \pounds_v \Pi_{jr}^s - T_{js}^i \pounds_v \Pi_{kr}^s - (\dot{\partial}_s T_{jk}^i)(\pounds_v \Pi_{tr}^s)\dot{x}^t \quad \text{... (2.4)}$$

and $$\nabla_j(\pounds_v \Pi_{kh}^i) - \nabla_k(\pounds_v \Pi_{jh}^i) = (\pounds_v N_{jkh}^i) + (\dot{\partial}_r \Pi_{kh}^i)(\pounds_v \Pi_{tj}^r)\dot{x}^t - (\dot{\partial}_r \Pi_{jh}^i)(\pounds_v \Pi_{tk}^r)\dot{x}^t \quad \text{... (2.5)}$$

where, N_{jkh}^i is the normal projective curvature tensor and this curvature tensor satisfies the identities and contractions have been given as

$$v^p N_{kjp}^i = (\rho_k \delta_j^i - \rho_j \delta_k^i) + \rho(\delta_k^i \emptyset_j - \delta_j^i \emptyset_k) + (\partial_k \emptyset_j - \partial_j \emptyset_k)v^i . \quad \text{... (2.6)}$$

Now we give the following definitions:

Definition (2.1): A Finsler space F_n is defines to be an affine motion if there exists a vector field $v^i(x)$ satisfying

$$\pounds_v \Pi_{jk}^i = 0. \quad \text{... (2.7)}$$

Definition (2.2): A Finsler space F_n is defines to be an affinely connected if

$$\dot{\partial}_r \Pi_{jk}^i = 0. \quad \text{...(2.8)}$$

Definition (2.3): The infinitesimal point transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines to a normal projective curvature collineation in a Finsler space F_n if

$$\pounds_v N_{jkh}^i = 0. \quad \text{... (2.9)}$$

Definition (2.4): A Finsler space F_n is defines to a Ricci normal projective curvature collineation provided there exists a vector field $v^i(x)$ satisfying

$$\pounds_v N_{kh} = 0. \quad \text{... (2.10)}$$

Definition (2.5): The infinitesimal point transformation $\bar{x}^i = x^i + v^i(x)dt$ is said to be an infinitesimal normal projective transformation in an F_n if

$$\pounds_v \Pi_{jk}^i = \delta_j^i b_k + \delta_k^i b_j + g_{jk} g^{ir} d_r \quad \text{... (2.11)}$$

where $b_j(x,\dot{x})$ and $d_r(x,\dot{x})$ are vector fields satisfying the following

(a) $\dot{\partial}_j b = b_j$, (b) $\dot{\partial}_k b_j = b_{jk}$, (c) $b_{jk}\dot{x}^k = b_j$, ... (2.12)

(d) $b_j \dot{x}^j = b$, (e) $\dot{\partial}_j d = d_j$, (f) $\dot{\partial}_k d_j = d_{jk}$,

(g) $d_{jk}\,\dot{x}^k = d_j$ and (h) $d_j\,\dot{x}^j$ = d .

Using (2.9) in (2.5), we get

$$\pounds_v N^i_{jkh} = \delta^i_k(\nabla_j\, b_h) + \delta^i_h(\nabla_j\, b_k) + g_{kh}\, g^{ir}(\nabla_j\, d_r) - \delta^i_j(\nabla_k\, b_h)$$

$$- \delta^i_k(\nabla_h\, b_j) - g_{jh}\, g^{ir}(\nabla_k\, d_r) - b\,(\dot{\partial}_j \Pi^i_{kh}) - (\dot{\partial}_m \Pi^i_{kh})\, g_{rj}\, g^{rp}\, d_p\, \dot{x}^m. \quad \text{... (2.13)}$$

where, Kroncker delta and the two fundamental tensors have vanishing normal projective covariant derivatives.

Allowing a transvection of (2.13) by $\dot{x}^j\,\dot{x}^k$, we have

$$\pounds_v N^i_{jkh}\; \dot{x}^j\,\dot{x}^k = \delta^i_k(\nabla_j\, b_h)\dot{x}^j\,\dot{x}^k + \delta^i_h(\nabla_j\, b)\dot{x}^j + g_{kh}\, g^{ir}(\nabla_j\, d_r)\dot{x}^j\,\dot{x}^k$$

$$- \delta^i_h(\nabla_k\, b)\dot{x}^k - g_{jh}\, g^{ir}(\nabla_k\, d_r)\dot{x}^j\,\dot{x}^k - (\dot{\partial}_r \Pi^i_{kh}) g_{sj}\, g^{rp}\, d_p\, \dot{x}^s\dot{x}^j\,\dot{x}^k\;, \quad \text{... (2.14)}$$

where, we have taken into account (2.12).

Now allowing a contraction in (2.14) with respect to the indices i and h and get

$$\pounds_v N^i_{jki}\; \dot{x}^j\,\dot{x}^k = (n+1)(\nabla_j\, b)\dot{x}^j - (n+1)(\nabla_k\, b)\dot{x}^k + (\nabla_j\, d)\dot{x}^j - (\nabla_k\, d)\dot{x}^k$$

$$- (\dot{\partial}_r \Pi^i_{kh}) g_{sj}\, g^{rp}\, d_p\, \dot{x}^s\dot{x}^j\,\dot{x}^k\;. \quad \text{... (2.15)}$$

Now allowing contraction (2.13) with respect to the indices i and j and thereafter using (2.6) we get

$$\pounds_v N_{kh} = \delta^i_k(\nabla_i\, b_h) + \delta^i_h(\nabla_i\, b_k) + g_{kh}\, g^{ir}(\nabla_i\, d_r) - \mathrm{n}(\nabla_k\, b_h)$$

$$- \delta^i_k(\nabla_h\, b_i) - \delta^r_h(\nabla_k\, d_r) - \mathrm{b}(\dot{\partial}_i \Pi^i_{kh}) - \delta^{\mathrm{p}}_i\, (\dot{\partial}_m \Pi^i_{kh}) d_p\, \dot{x}^m. \quad \text{... (2.16)}$$

We now eliminate the term $(\nabla_k b)\dot{x}^k$ with the help of (2.14) and (2.15), we get

$$Q^i_h = (n+1)[(\nabla_j\, b_h)\dot{x}^i\dot{x}^j + g_{kh}\, g^{ir}(\nabla_j\, d_r)\dot{x}^j\,\dot{x}^k - (\nabla_k\, b_h)\dot{x}^i\,\dot{x}^k$$

$$- g_{jh}\, g^{ir}(\nabla_k\, d_r)\dot{x}^j\,\dot{x}^k - (\dot{\partial}_r \Pi^i_{kh}) g_{sj}\, g^{rp}\, d_p\, \dot{x}^s\dot{x}^j\,\dot{x}^k]$$

$$+ (\dot{\partial}_r \Pi^i_{kh}) g_{sj}\, g^{rp}\, d_p\, \dot{x}^s\dot{x}^j\,\dot{x}^k\;. \quad \text{... (2.17)}$$

where $Q^i_h = (\mathrm{n}+1)\, \pounds_v N^i_{jkh}\; \dot{x}^j\,\dot{x}^k - \delta^i_h\, \pounds_v N^i_{jki}\; \dot{x}^j\,\dot{x}^k$.

We now take into account the projective deviation tensor $W^i_j(x, \dot{x})$ as has been given

$$W^i_j = H^i_j - \mathrm{H}\,\delta^i_j - \frac{1}{n+1}(\dot{\partial}_k H^k_j - \dot{\partial}_j \mathrm{H})\,\dot{x}^i$$

apply the commutation formula (2.4) to this deviation tensor and get

$$\pounds_v(\nabla_r W^i_j) - \nabla_r(\pounds_v W^i_j) = W^s_j \pounds_v \Pi^i_{sr} - W^i_s\, \pounds_v \Pi^s_{jr} - (\dot{\partial}_s W^i_j)(\pounds_v \Pi^s_{tr})\dot{x}^t\;. \quad \text{... (2.18)}$$

Using (2.11) in (2.18), we get

$$\pounds_v(\nabla_r W_j^i) - \nabla_r(\pounds_v W_j^i) = W_j^s(\delta_s^i b_r + \delta_r^i b_s + g_{sr} g^{ip} d_p) - W_s^i(\delta_j^s b_r + \delta_r^s b_j$$
$$+ g_{jr} g^{sp} d_p) - (\dot{\partial}_s W_j^i)(\delta_t^s b_r + \delta_r^s b_t + g_{tr} g^{sp} d_p)\dot{x}^t \text{ . ... (2.19)}$$

Now we make the assumption that $\pounds_v(\nabla_r W_j^i) = 0$, then from (2.19), we have

$$\nabla_r(\pounds_v W_j^i) = (\dot{\partial}_s W_j^i) b_r \dot{x}^s + \mathrm{b}(\dot{\partial}_r W_j^i) + (\dot{\partial}_s W_j^i)\, g_{tr} g^{sp} d_p \dot{x}^t + W_r^i\, b_j$$
$$+ W_s^i\, g_{jr} g^{sp} d_p - W_j^s \delta_r^i b_s - W_j^s\, g_{sr} g^{ip} d_p \text{ . ... (2.20)}$$

We now allowing a contraction in (2.20) with respect to the indices i and r and, get

$$\nabla_i(\pounds_v W_j^i) = (\dot{\partial}_s W_j^i) b_i \dot{x}^s + (\dot{\partial}_s W_j^i) g_{ti} g^{sp} d_p \dot{x}^t + W_s^i g_{ji} g^{sp} d_p - \mathrm{n}\, W_j^s b_s - W_j^s d_s \text{ .... (2.21)}$$

Allowing a transvection in (2.20) by $\dot{x}^r$, we get

$$\nabla_r(\pounds_v W_j^i)\dot{x}^r = (\dot{\partial}_s W_j^i) b\, \dot{x}^s + b\,(\dot{\partial}_r W_j^i)\dot{x}^r + (\dot{\partial}_s W_j^i)\, g_{tr} g^{sp} d_p \dot{x}^t \dot{x}^r + W_r^i b_j \dot{x}^r$$
$$+ W_s^i\, g_{jr} g^{sp} d_p \dot{x}^r - W_j^s b_s \dot{x}^r - W_j^s\, g_{sr} g^{ip} d_p \dot{x}^r \text{ , ... (2.22)}$$

while writing (2.22), we have taken into account

(a) $W_j^i \dot{x}^r = 0$, (b) $\dot{\partial}_h W_r^i\, \dot{x}^r = -W_h^i$, (c) $\dot{\partial}_i W_j^i = 0$,

(d) $\dot{\partial}_j W_k^i\, \dot{x}^j = 2W_k^i$, (e) $W_i^i = 0$. ... (2.23)

We now eliminating the term $W_j^s b_s$ with the help of (2.21) and (2.22) and after get

$$\mathrm{n}\, \nabla_r(\pounds_v W_j^i)\dot{x}^r - \nabla_r(\pounds_v W_j^i)\dot{x}^r = R_j^i + S_j^i \text{ , ... (2.24)}$$

where $R_j^i = n\,[(\dot{\partial}_s W_j^i) b\, \dot{x}^s + b\,(\dot{\partial}_r W_j^i)\dot{x}^r + (\dot{\partial}_s W_j^i)\, g_{tr} g^{sp} d_p \dot{x}^t \dot{x}^r$
$$- W_j^s\, g_{sr} g^{ip} d_p \dot{x}^r] \text{ ... (2.25)}$$

and $$S_j^i = [(\dot{\partial}_s W_j^i) b\, \dot{x}^s + (\dot{\partial}_s W_j^i)\, g_{tr} g^{sp} d_p \dot{x}^r \dot{x}^t - W_s^t g_{jt} g^{sp} d_p \dot{x}^i] \text{ . ... (2.26)}$$

Now in view of given definitions we shall summarize all those observations and we have obtained the result in the form of theorems. Keeping in mind the definition (2.1) and given by (2.6), from (2.11) we arrive at the conclusion that if the infinitesimal transformation $\bar{x}^i = x^i + v^i\,(x)dt$ defines an affine motion then the covariant vector fields $b_j(x, \dot{x})$ and $d_r(x, \dot{x})$ should vanish. Therefore, we can state.

Theorem (2.1): If the infinitesimal normal projective transformation $\overline{x}^i = x^i + v^i (x)dt$ is defines an affine motion in a Finsler space F_n , then the covariant vector fields $b_j(x, \dot{x})$ and $d_r(x, \dot{x})$ appearing in (2.11) should vanish.

Now we consider that the infinitesimal transformation $\bar{x}^i = x^i + v^i\,(x)dt$ defines a normal projective curvature collineation then in this case with the help of (2.9) and (2.13), we can state the following:

Theorem (2.2): If the infinitesimal normal projective point transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines a normal projective curvature collineation in a Finsler space F_n, then we shall have

$$\delta^i_k(\nabla_j b_h) + \delta^i_h(\nabla_j b_k) + g_{kh}\, g^{ir}(\nabla_j d_r) - \delta^i_j(\nabla_k b_h) - \delta^i_k(\nabla_h b_j)$$

$$- g_{jh}\, g^{ir}\,(\nabla_k d_r) - b(\dot{\partial}_j \Pi^i_{kh}) - (\dot{\partial}_m \Pi^i_{kh}) g_{rj}\, g^{rp} d_p \dot{x}^m = 0 . \qquad \text{... (2.27)}$$

Now if we assume that the covariant vector fields $b_j(x, \dot{x})$ and $d_r(x, \dot{x})$ appearing in (2.11) are normal projective covariant constants then in this case with the help of (2.13) we can state:

Theorem (2.3): **If the covariant vector fields $b_j(x, \dot{x})$ and $d_r(x, \dot{x})$ appearing in (2.11) are projective covariant constants in a Finsler space F_n, then we have**

$$\pounds_v N^i_{jkh} + b(\dot{\partial}_j \Pi^i_{kh}) + (\dot{\partial}_m \Pi^i_{kh}) g_{rj} g^{rp} d_p \dot{x}^m = 0 . \qquad \text{... (2.28)}$$

If we assume that in the case of normal projective curvature collineation then we get:

$$b(\dot{\partial}_j \Pi^i_{kh}) + (\dot{\partial}_m \Pi^i_{kh}) g_{rj}\, g^{rp}\, d_p\, \dot{x}^m = 0 \qquad \text{... (2.29)}$$

where, the covariant vector fields $b_j(x, \dot{x})$ and $d_r(x, \dot{x})$ appearing in (2.11) be assumed to be normal projective covariant constants.

Now we consider the case when the infinitesimal point transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines a normal projective Ricci collineation then in this case we can state with the help of (2.10) and (2.16):

Theorem (2.4): If the infinitesimal normal projective transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines a normal projective Ricci collineation in a Finsler space F_n, then we have

$$(1-n)\,(\nabla_k b_h) + g_{kh}\, g^{ir}(\nabla_j d_r) - (\nabla_k b_h) - b\,(\dot{\partial}_i \Pi^i_{kh}) - (\dot{\partial}_m \Pi^i_{kh})\, d_i \dot{x}^m = 0 . \qquad \text{... (2.30)}$$

Now, in a Finsler space if we assume that the infinitesimal point transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines non-affine such that the normal projective covariant derivative of the deviation tensor W^i_j is an invariant then, we can state:

Theorem (2.5): If the infinitesimal normal projective transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines non-affine in a Finsler space F_n, then (2.18) is always true provided the considered infinitesimal normal projective transformation leaves invariant the projective covariant derivative of the deviation tensor W^i_j.

Whereas, in case the infinitesimal point transformation is affine one then we can state:

Theorem (2.6): If the infinitesimal normal projective transformation $\bar{x}^i = x^i + v^i(x)dt$ is affine one in a Finsler space F_n, then $Q_h^i = 0$, provided the considered infinitesimal transformation leaves invariant the projective covariant derivative of the deviation tensor W_j^i . where

$$Q_h^i = (n+1)\,\pounds_v N_{jkh}^i\, \dot{x}^j \dot{x}^k - \delta_h^i\, \pounds_v N_{jki}^i\, \dot{x}^j \dot{x}^k . \qquad ... (2.31)$$

We now consider the case when symmetric Finsler space, then the normal projective covariant derivative of the deviation tensor W_j^i vanishes i.e. $\nabla_k W_j^i = 0$, then, we can state:

Theorem (2.7): If the infinitesimal normal projective transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines non-affine one in a symmetric Finsler space F_n, then (2.23) always holds where R_j^i and S_j^i appearing in (2.24) have been given by (2.25) and (2.26) respectively.

Theorem (2.8): If assume that the infinitesimal normal projective transformation $\bar{x}^i = x^i + v^i(x)dt$ is deffines affine one in a symmetric Finsler space F_n, then $Q_h^i = 0$ always holds, where Q_h^i has been given by (2.31).

INFINITESIMAL PROJECTIVE TRANSFORMATION WITH SPECIAL REFERENCE TO CARTAN'S CONNECTION $\Gamma_{jk}^{*i}(x, \dot{x})$:

The studies in this section keeping in mind the Cartan's first covariant derivative has been defined as

$$T_{ij\,|h} = \partial_h T_{ij} - \dot{\partial}_k T_{ij}\, \dot{\partial}_h G^k - T_{kj}\, \Gamma_{ih}^{*k} - T_{ik} \Gamma_{jh}^{*k} , \qquad ... (3.1)$$

the Lie-derivatives of an arbitrary tensor $T_j^i(x, \dot{x})$ and Cartan's connection coefficients $\Gamma_{jk}^{*i}(x, \dot{x})$ can be written in the following forms by Yano [10],

$$\pounds_v T_j^i(x, \dot{x}) = T_{j|r}^i\, v^s + (\dot{\partial}_t\, T_j^i) v_{|s}^t\, \dot{x}^r - T_j^s v_{|s}^i + T_s^i v_{|j}^s \qquad ... (3.2)$$

and
$$\pounds_v \Gamma_{jk}^{*i}(x, \dot{x}) = v_{|jk}^i + K_{jks}^i\, v^s + (\dot{\partial}_s\, \Gamma_{jk}^{*i}) v_{|t}^s\, \dot{x}^t , \qquad ... (3.3)$$

where, the quantities K_{jks}^i is Cartan's first curvature tensor defined by

$$K_{jks}^i\, \dot{x}^j = \partial_s G_k^i - \partial_k G_s^i - G_{kr}^i\, G_s^r + G_{sr}^i\, G_k^r , \qquad ... (3.4)$$

and this curvature tensor satisfies the identities as given by following forms,

(a) $K_{jks}^i = -K_{jsk}^i$, (b) $K_{jks}^i + K_{ksj}^i + K_{sjk}^i = 0$. ... (3.5)

The following commutation formulae involving the Lie derivative and Cartan's first covariant derivative as given by (Yano [10])

$$\pounds_v(\dot{\partial}_r T_j^i) - \dot{\partial}_r(\pounds_v T_j^i) = 0\ , \quad \text{... (3.6)}$$

$$(\pounds_v T_j^i)_{|s} - \pounds_v(T_{j|s}^i) = T_j^t\, \pounds_v \Gamma_{st}^{*i} - T_t^i \pounds_v \Gamma_{js}^{*t} - (\dot{\partial}_t\, T_j^i)\ \pounds_v \Gamma_{sm}^{*t}\ \dot{x}^m \quad \text{... (3.7)}$$

$$(\pounds_v \Gamma_{jk}^{*i})_{|r} - \pounds_v(\Gamma_{jk|r}^{*i}) = \pounds_v K_{jkr}^i + 2(\dot{\partial}_s\ \Gamma_{j[k}^{*i})\ \pounds_v \Gamma_{r]t}^{*s}\dot{x}^t\ . \quad \text{... (3.8)}$$

Now we defined the following definitions which are use in the later discussion:

Definition (3.1): A Finsler space F_n is defines to be an affine motion if there exists a vector field $v^i(x)$ satisfying

$$\pounds_v \Gamma_{jk}^{*i} = 0. \quad \text{... (3.9)}$$

Definition (3.2): A Finsler space F_n is defines to be an affinely connected if

$$\dot{\partial}_r\ \Gamma_{jk}^{*i} = 0. \quad \text{... (3.10)}$$

Definition (3.3): The infinitesimal point transformation $\bar{x}^i = x^i + v^i\ (x)dt$ is defines to a Cartan's curvature collineation in a Finsler space F_n if

$$\pounds_v K_{jhk}^i = 0. \quad \text{... (3.11)}$$

Definition (3.4): A Finsler space F_n is defines to a Cartan's Ricci collineation provided there exists a field $v^i(x)$ satisfying

$$\pounds_v K_{jk} = 0. \quad \text{...(3.12)}$$

Difinition (3.5): The infinitesimal point transformation $\bar{x}^i = x^i + v^i\ (x)dt$ is defines an infinitesimal projective transformation in a Finsler space F_n , if

$$\pounds_v \Gamma_{jk}^{*i} = \delta_j^i b_k + \delta_k^i b_j - g_{jk}\, g^{ir}\, d_r\ , \quad \text{... (3.13)}$$

where $b_j(x, \dot{x}\ )$ and $d_r(x, \dot{x}\ )$ appearing in (3.13) satisfies the identities as given in (2.12).

Using (3.13) in the commutation formula given by (3.8), we get

$$\pounds_v K_{hjk}^i = 2\{\delta_h^i b_{[jk\,]} + b_{h|[k}\delta_{j]}^i + g^{ir}\, d_{r|[k}\, g_{j\,]h} - \dot{\partial}_s \Gamma_{h[j}^{*i}\ g_{k]t}\, g^{sr}\, d_r\ \dot{x}^t\}. \quad \text{... (3.14)}$$

Now we allowing a transaction in (3.14) by $\dot{x}^h$ and $\dot{x}^j$ and get

$$\pounds_v K_{hjk}^i\ \dot{x}^h\ \dot{x}^j = 2b_{|k}\dot{x}^i \text{-}\ \delta_k^i b_{|h}\dot{x}^h - b_{k|j}\,\dot{x}^i\dot{x}^j - g^{ir}\,\dot{x}^h\ \dot{x}^j\,(g_{hj}\, d_{r|k} - g_{hk}\, d_{r|j}) \text{....} \quad (3.15)$$

Now we allowing a contraction in (3.14) with respect to the indices i and k and get

$$\pounds_v K_{hj} = b_{j\,|h} - n\, b_{h|j} + d_{h|j} - d_r\, g^{sr}\{(\dot{\partial}_s \Gamma_{hj}^{*i})g_{ti} - (\dot{\partial}_s \Gamma_{hi}^{*i})g_{tj}\}\dot{x}^t, \quad \text{... (3.16)}$$

where $\qquad K_{hj} = K_{hji}^i\,.$

Now we allow a transvection in (3.16) by $\dot{x}^h\ \dot{x}^j$ and thereafter use (2.11) and get

$$\pounds_v K_{hj}\ \ \dot{x}^h\ \dot{x}^j = \{(1\text{–}n)b_{|h} + d_{|h}\ \}\dot{x}^h. \quad \text{... (3.17)}$$

Now eliminate the term $b_{|h}\dot{x}^h$ using (3.15) and (3.17), we get

$$M_j^i(x,\dot{x}) = \delta_k^i d_{|h}\dot{x}^h + (1-n)\{\dot{x}^i(2b_{|k} - b_{k|j}\dot{x}^j) - g^{ir}\dot{x}^h\dot{x}^j(g_{hj}d_{r|k} - g_{hk}d_{r|j})\}, \quad ...(3.18)$$

where $M_k^i(x,\dot{x}) = (1-n)\pounds_v K_{hjk}^i\ \dot{x}^h\ \dot{x}^j + \pounds_v K_{hj}\ \dot{x}^h\ \dot{x}^j\delta_k^i$ . ... (3.19)

Now we take the projective deviation tensor $W_j^i(x,\dot{x})$ as has been given

$$W_j^i = H_j^i - \mathrm{H}\,\delta_j^i - \frac{1}{n+1}(\dot{\partial}_i H_j^i - \dot{\partial}_j \mathrm{H})\,\dot{x}^i , \quad ... (3.20)$$

and applying the commutation formula (3.7) and get

$$(\pounds_v W_j^i)_{|s} - \pounds_v W_{j|s}^i = W_j^h\,\pounds_v\Gamma_{hs}^{*i} - W_h^i\,\pounds_v\Gamma_{js}^{*h} - (\dot{\partial}_h W_j^i)\pounds_v\Gamma_{st}^{*h}\dot{x}^t. \quad ... (3.21)$$

Using (3.8) in (3.21), we get

$$(\pounds_v W_j^i)_{|s} - \pounds_v W_{j|s}^i = b_h W_j^h\delta_s^i - b_j W_s^i - 2b_s W_j^i - \mathrm{b}\,(\dot{\partial}_s W_j^i)$$
$$- d_r[W_j^h g_{hs}g^{ir} - g^{hr}\{W_h^i g_{js} + (\dot{\partial}_h W_j^i)\,g_{st}\dot{x}^t\}], \quad ... (3.22)$$

while (3.22), we have taken into account as given in (2.11).

Now we make the Cartan's first covariant derivative of the projective deviation tensor is a Lie invariant i.e. $\pounds_v W^i_{j|s} = 0$, then from (3.22), we get

$$(\pounds_v W_j^i)_{|s} = b_h W_j^h\delta_s^i - b_j W_s^i - 2b_s W_j^i - \mathrm{b}\,(\dot{\partial}_s W_j^i)$$
$$- d_r[W_j^h g_{hs}g^{ir} - g^{hr}\{W_h^i g_{js} + (\dot{\partial}_h W_j^i)\,g_{st}\dot{x}^t\}]. \quad ... (3.23)$$

Now we allowing a contraction in (3.23) with respect to the indices i and s, we get

$$(\pounds_v W_j^i)_{|i} = (n-2)\,b_h W_j^h - d_h W_j^h + g^{hr}d_r\{W_h^s g_{js} + (\dot{\partial}_h W_j^i)\,g_{it}\dot{x}^t\}. \quad ...(3.24)$$

Now allowing a transvection in (3.23) by $\dot{x}^s$, we get

$$(\pounds_v W_j^i)_{|s}\dot{x}^s = b_h W_j^h\dot{x}^i - 4bW_j^i - d_r\dot{x}^s[W_j^h g_{hs}g^{ir}$$
$$- g^{hr}\{W_h^i g_{js} + (\dot{\partial}_h W_j^i)\,g_{st}\dot{x}^t\}] , \quad ... (3.25)$$

while writing (3.25), we have taken into account (2.23).

Now eliminating the term $b_h\ W_j^h$ with the help of (3.24) and (3.25), we get

$$(n\text{-}2)\,(\pounds_v W_j^i)_{|s}\dot{x}^s - (\pounds_v W_j^i)_{|s}\dot{x}^s = E_j^i - F_j^i \quad ... (3.26)$$

where $E_j^i = W_j^h d_h\dot{x}^i - g^{hr}d_r\dot{x}^i\{W_h^r g_{js} + (\dot{\partial}_h W_j^s)g_{st}\dot{x}^t\}$ ... (3.27)

and $F_j^i = (n-2)[4bW_j^i + d_r\dot{x}^s\{W_j^h g_{hs}g^{ir} - g^{hr}W_h^i g_{js} + g^{hr}(\dot{\partial}_h W_j^i)\,g_{st}\dot{x}^t\}]$ ....(3.28)

Now we summaries all these observations in the light of definitions given in this section and accordingly statements can be made in the form of theorems. we take into account the definition (3.5), we arrive at the results if the infinitesimal transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines an affine motion then in such a case the vector fields $b_j(x, \dot{x})$ and $d_r(x, \dot{x})$ should vanish and therefore we can state:

Theorem (3.1): If assume the infinitesimal transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines an affine motion in a Finsler space F_n, then the covariant vector fields $b_j(x, \dot{x})$ and $d_r(x, \dot{x})$ appearing in (3.13) should vanish.

Now we consider when the infinitesimal point transformation $x^i + v^i(x)dt$ defines a Cartan's curvature collineation then in such a case with the help of (3.11) and (3.14), we get

$$\delta_h^i b_{[jk]} + b_{h|[k}\delta_{j]}^i + g^{ir} d_{r|[k} g_{j]h} - \dot{\partial}_s \Gamma_{h[j}^{*i} g_{k]t} g^{sr} d_r \dot{x}^t = 0. \qquad \text{... (3.29)}$$

With the help of (3.29), therefore we can state:

Theorem (3.2): If the infinitesimal transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines Cartan's curvature collineation in a Finsler space F_n, then (3.29) always holds.

Now if we assume that the vector fields $b_j(x, \dot{x})$ and $d_r(x, \dot{x})$ are covariant constants with respect to Cartan's first covariant derivative then, we can state:

Theorem (3.3): If the covariant vector fields $b_j(x, \dot{x})$ and $d_r(x, \dot{x})$ appearing in (3.13) be defined to be covariant constants with respect to Cartan's first covariant derivative then, we have

$$\pounds_v K_{hjk}^i + \dot{\partial}_s \Gamma_{h[j}^{*i} g_{k]t} g^{sr} d_r \dot{x}^t = 0. \qquad \text{...(3.30)}$$

Now we consider when the infinitesimal transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines Cartan's Ricci collineation then in such case with the help of (3.10) and (3.16), we can state:

Theorem (3.4): If the infinitesimal transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines Cartan's Ricci collineation in a Finsler space F_n, then we have

$$b_{j|h} - n b_{h|j} + d_{h|j} - d_r g^{sr}\{(\dot{\partial}_s \Gamma_{hj}^{*i}) g_{ti} - (\dot{\partial}_s \Gamma_{hi}^{*i}) g_{tj}\}\dot{x}^t = 0. \qquad \text{...(3.31)}$$

Now, we assume that the infinitesimal point transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines non-affine such that the Cartan's first covariant derivative of the deviation tensor W_j^i, is an invariant then under such case we can state:

Theorem (3.5): If the infinitesimal transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines non-affine one in a Finsler space F_n, then (3.21) is always true, provided that the assumption infinitesimal transformation leaves invariant the Cartan's first covariant derivative of the deviation tensor.

Now suppose when the infinitesimal transformation is affine one then in such case, we can state:

Theorem (3.6): If the infinitesimal transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines affine one in a Finsler space F_n , then $M_k^i = 0$ provided that the assumption infinitesimal transformation leaves invariant the Cartan's first covariant derivative of the deviation tensor where $M_k^i(x, \dot{x})$ is given by (3.19).

Now we assume that when the case is symmetric Finsler space, then the Cartan's first covariant derivative of the deviation tensor will vanish i.e. $W_{j|k}^i = 0$, we can state:

Theorem (3.7): **If the infinitesimal transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines non-affine one in a symmetric Finsler space F_n , then (3.26) is always true.**

Theorem (3.8): If the infinitesimal transformation $\bar{x}^i = x^i + v^i(x)dt$ is defines affine one in a symmetric Finsler space F_n , then (3.27) is always true.

CONCLUSION :

This paper has been devoted to the study of Infinitesimal transformation in a Finsler space. The paper has been divided into three sections of which the first section is introductory and the second section deals with normal projective infinitesimal transformation and curvature collineation in a Finsler space F_n. After giving a series of definitions we obtained the results involving Lie-derivatives of normal projective curvature tensor N_{jkh}^i and the Ricci tensor N_{kh} and in this continuation we have derived many more results involving the projective deviation tensor and its Lie derivatives and consequent upon these derivations we have put our observations on record in the form of theorems telling as to what will happen to the covariant vector fields $b_j(x, \dot{x})$ and $d_r(x, \dot{x})$ when the infinitesimal normal projective transformation $\bar{x}^i = x^i + v^i(x)dt$ defines an affine as well as non-affine motion, and in continuation have also derived the results which will hold in case the infinitesimal normal projective transformation under consideration defines normal projective curvature and normal projective Ricci collineation and in the last results have been derived when the infinitesimal transformation under consideration is itself non-affine and affine one respectively. The third and the last section we have study of infinitesimal projective transformation with special reference to Cartan's connection $\Gamma_{jk}^{*i}(x, \dot{x})$ here the previous section we have taken in the form of normal projective connection $\Pi_{jk}^i(x, \dot{x})$, the two connection coefficients are quite different so results will be different. After carrying out a series of related derivations we have summarized the observations in the form of theorems notable amongst them are as to what will happen to the covariant vector fields $b_j(x, \dot{x})$ and $d_r(x, \dot{x})$ when the infinitesimal transformation under the consideration defines an affine motion and also as to what relationships will hold in such a case when the covariant vectors are assumed to be covariant constants. In this continuation we have also derived the relationships which will hold when the infinitesimal transformation under consideration defines Cartan's curvature collineation and Cartan's Ricci collineation. In the last we have derived the relationships which will hold in a symmetric Finsler space when the infinitesimal transformation under consideration is affine and non-affine one.

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