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SOME CHARACTERISTIC PROPERTIES OF FUZZY IDEALS ON A BCH-ALGEBRA

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In this paper we have established some characteristic properties of fuzzy ideals on some specific extended BCH- algebras.

Key words:- Fuzzy Ideal, BCH-algebra, Disjoint Elements.

INTRODUCTION

Definition (1.1) : (a) A system $(X; *, 0)$ consisting of a non-empty set X, a binary operation '$*$' and a fixed element 0 is called a BCH-algebra **[3]** if the following conditions are satisfied:

(BCH1) $x * x = 0$

(BCH2) $x * y = 0 = y * x \Rightarrow x = y$

(BCH3) $(x * y) * z = (x * z) * y$

for all $x, y, z \in X$.

(b) If in addition to (BCH1), (BCH2), (BCH3) the condition

(BCH4) $0 * x = 0$, for all $x \in X$

is also satisfied then $(X; *, 0)$ is a positive BCH-algebra.

Definition (1.2):- A pair $\{x, y\}$ of distinct elements of X is said to be mutually disjoint **[7]** if $x * y = x$ and $y * x = y$.

Now we mention some extensions of BCH- algebras which appear in **[4]** from a given BCH- algebra as follows:

Theorem (1.3):- Let $(X; *, 0)$ be a BCH-algebra and $t \notin X$. Let $Y = X \cup \{t\}$.

We define a binary operation '$\otimes$' in Y as

$$x \otimes y = x * y \text{ if } x, y \in X, \qquad ...(1.1)$$

$x \otimes t = x$ for $x \neq 0$; $x \in X$ and $t \otimes x = t$; $x \in X$, ...(1.2)

$0 \otimes t = t$, $t \otimes 0 = t$, $t \otimes t = 0$ ...(1.3)

Then $(Y; \otimes, 0)$ is a BCH-algebra iff X is a positive BCH-algebra.

Corollary (1.4):- In case X is not a positive BCH-algebra then taking $0 \otimes t = 0$ in (1.3), $(Y; \otimes, 0)$ becomes a BCH-algebra.

Theorem (1.5):- Every BCH-algebra $(X; *, 0)$ with a pair of non-zero mutually disjoint elements u and v can be extended to a BCH-algebra $Y = X \cup \{b\}, b \notin X$ under the binary operation '$\otimes$' defined as

$x \otimes y = x*y$ if $x, y \in X$, ...(1.4)

$x \otimes b = x$ if $x \neq u, x \neq v$, $x \in X$, ...(1.5)

$u \otimes b = b$ and $v \otimes b = 0$, ...(1.6)

$0 \otimes b = 0$, $b \otimes 0 = b$, $b \otimes b = 0$, ...(1.7)

$b \otimes u = 0$, $b \otimes y = b$, $y \neq u$ ...(1.8).

Theorem (1.6):- Let $(X; *, 0)$ be a BCH- algebra and let $Y = X \times X$. For $u, v \in Y$ with $u = (x_1, x_2)$, $v = (y_1, y_2)$ a binary operation '$\odot$' is defined in Y as

$u \odot v = (x_1 * y_1, x_2 * y_2)$ ...(1.9)

where $x_1, x_2, y_1, y_2 \in X$. Further we put $\tilde{0} = (0, 0)$.

Then $(Y; \odot, \tilde{0})$ is a BCH-algebra.

Definition (1.7):- A fuzzy set in X is a function

$\mu : X \rightarrow [0, 1]$.

Definition (1.8):-A fuzzy subset μ of a BCH-algebra X is said to be a fuzzy ideal of X if it satisfy:

$\mu(0) \geq \mu(x)$ for all $x \in X$

$\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

Example (1.9) :- Let $0 < c < 1$. Then every constant function with value c is a fuzzy ideal of X.

SOME RESULTS ON FUZZY IDEALS ON BCH-ALGEBRA

Lemma (2.1):- If μ is a fuzzy ideal on BCH- algebra X then $x \leq y \Rightarrow \mu(y) \leq \mu(x)$.

Proof :- We have $x \leq y \Rightarrow x * y = 0$.

Now $\mu(\mathrm{x}) \geq \min \{\mu(x * y), \mu(y)\} = \min \{\mu(0), \mu(y)\} = \mu(y)$

$$\Rightarrow \mu(y) \leq \mu(x).$$

Theorem (2.2) :- Let μ be a fuzzy set defined on a BCH-algebra X such that $\mu(0) \geq \mu(x)$ for all $x \in X$. Then μ is a fuzzy ideal on X iff for all $x, y, z \in X$ with $(x * y) * z = 0$,

$$\mu(x) \geq \min \{ \mu(y), \mu(z)\} \text{ satisfied.}$$

Proof :- Let μ be a fuzzy ideal on X. Then

$$\mu(x * y) \geq \min \{\mu((x * y) * z), \mu(z)\}$$

$$= \min\{\mu(0), \mu(\mathrm{z})\}$$

$$= \mu(z).$$

Again since $(x * y) * z = 0 \Longrightarrow (x * z) * y = 0$, we get

$$\mu(x * z) \geq \mu(y),$$

Now $\mu(x) \geq \min \{ \mu(x * z), \mu(z)\}$.

If $\mu(\mathrm{x} * z) \leq \mu(z)$ then $\mu(x) \geq \mu(x * z) \geq \mu(y)$.

If $\mu(x * z) \geq \mu(z)$ then $\mu(x) \geq \mu(z)$. Combining these facts we get

$$\mu(x) \geq \min\{\mu(y), \mu(z)\}.$$

Conversely, suppose that the given condition is satisfied.

We put $z = x * y$.

Then $(x * y) * z = 0$.

So $\mu(x) \geq \min\{\mu(y), \mu(z)\}$

$$= \min\{\mu(y), \mu(x * y)\}.$$

Hence μ be a fuzzy ideal.

Theorem (2.3):- Let μ be a fuzzy ideal on BCH-algebra X. For any $t \notin X$, let $Y = X \cup \{t\}$. Let Y be extended BCH-algebra with binary operation '$\otimes$' as given in theorem (1.3). Then μ can be extended to a fuzzy ideal μ^1 defined on Y as $\mu^1(x) = \mu(x)$ if $x \in X$ and $\mu^1(\mathrm{t}) = \min\{ \mu(\mathrm{x}) ; \mathrm{x} \in X\}$.

Proof :- For $x, y \in X$ we have nothing to prove. We take a point as t and other as $x \in X$.

Since $\mu^1(x \otimes t) = \mu^1(x) = \mu(x)$, we have $\min\{\mu^1(x \otimes t), \mu^1(t)\} = \mu^1(t)$,

So $\mu^1(x) \geq \min\{\mu^1(x \otimes t), \mu^1(t)\} = \mu^1(t)$ is satisfied.

Now $\min\{\mu^1(t \otimes x), \mu^1(x)\} = \min\{\mu^1(t), \mu^1(x)\} = \mu^1(\mathrm{t})$

Also $\{\mu^1(t \otimes t), \mu^1(t)\} = \{\mu^1(0), \mu^1(t)\} = \mu^1(t)$

So $\mu^1(\mathrm{x}) \geq \min\{\mu^1(x \otimes t), \mu^1(t)\}$ and $\mu^1(t) \geq \min\{\mu^1(t \otimes x), \mu^1(x)\}$ are satisfied. Hence μ^1 is a fuzzy ideal on Y.

Definition (2.4):- Let X be a non- empty set and let μ be a fuzzy set in X. We consider the Cartesian product $X \times X$. Different types of fuzzy sets can be defined on $X \times X$. Let μ_1, μ_2, μ_3, μ_4 and μ_5 be fuzzy sets defined on $X \times X$ as follows:

For any $x, y \in X$, we define

(a) $\mu_1(x, y) = \mu(x)$ ... (2.1)

(b) $\mu_2(x, y) = \mu(y)$ ...(2.2)

(c) $\mu_3(x, y) = \min\{\mu(x), \mu(y)\}$ ... (2.3)

(d) $\mu_4(x, y) = \max\{\mu(x), \mu(y)\}$ ... (2.4)

(e) $\mu_5(x, y) = \mu(x)\cdot \mu(y)$ ... (2.5)

Some other fuzzy sets may be defined on $X \times X$.

Theorem (2.5):- Let X be a BCH- algebra and let μ be a fuzzy set defined on X. Let $Y = X \times X$. Then Y be a BCH–algebra under binary operation $\odot$ defined by (1.9) [Theorem (1.6)]. We consider fuzzy set μ_1 defined on Y by relation (2.1). Then μ is a fuzzy ideal on X iff μ_1 is fuzzy ideal on Y.

Proof :- Let μ be a fuzzy ideal defined on X. Then

(i) $\mu(0) \geq \mu(x)$ for all $x \in X$

(ii) $\mu(x) \geq \min\{\mu(x * y), \mu(\mathrm{y})\}$ for all $x, y \in X$.

Now (i) $\mu_1(0, 0) = \mu(0) \geq \mu(\mathrm{x}) = \mu_1(x, y)$ for all $(x, y) \in Y$. ...(2.6)

(ii) Let $(x, y) \in Y$. Then $\mu_1(x, y) = \mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.

This gives

$$\min\{\mu_1((x_1, y_1) \odot (x_2, y_2)), \mu_1(x_2, y_2)\}$$

$$= \min\{\mu_1(x_1 * x_2, y_1 * y_2), \mu_1(x_2, y_2)\}$$

$$= \min\{\mu(x_1 * x_2), \mu(\mathrm{x}_2)\} \leq \mu(x_1) = \mu_1(x_1, y_1)$$

i.e., $\mu_1(x_1, y_1) \geq \min\{\mu_1((x_1, y_1) \odot (x_2, y_2)), \mu_1(x_2, y_2)\}$ ... (2.7) for all $(x_1, y_1), (x_2, y_2) \in Y$.

This proves that μ_1 is fuzzy ideal on Y.

Conversely, suppose that μ_1 is fuzzy ideal on Y.

Then $\mu_1(0, 0) \geq \mu_1(x_1, y_1)$ for all $(x, y) \in Y$.

This implies $\mu(0) \geq \mu(x)$ for all $x \in X$.

Let $x, y \in X$. We consider $(x, 0), (y, 0) \in Y$.

From (2.7) we have $\mu_1(x, 0) \geq \min\{\mu_1((x, 0) \odot (y, 0)), \mu_1(y, 0)\}$

$$= \min\{\mu_1(x * y, 0), \mu_1(y, 0)\}$$

$$\mu(x) \geq \min\{\mu(x * y), \mu(y)\}.$$

So μ is a fuzzy ideal on BCH- algebra X.

Corollary (2.6):- Similar result can be given for μ_2 .

Theorem (2.8):- We recall X, Y, u, v, b and binary operation '$\otimes$' as given in theorem (1.5).

Let μ be a fuzzy ideal defined on X. We define μ^1 on Y as $\mu^1(x) = \mu(x)$ if $x \in X$ and $\mu^1(b) = \mu(u)$. Then μ^1 is a fuzzy ideal on Y iff $\mu(x) \geq \mu(u)$; for all $x \in X$.

Proof :- Suppose that μ^1 is a fuzzy ideal on Y then $\mu^1(0) \geq \mu^1(y)$ for all $y \in Y$, *i.e.*, for all $x \in X$.

Now for $x \in X$, $\mu^1(x) = \mu^1(x) \geq \min\{\mu^1(x \otimes b), \mu^1(b)\}$

$$= \min\{\mu(x), \mu(u)\}$$

$$\Rightarrow \quad \mu(x) \geq \mu(u) \text{ ; for all } x \in X.$$

So the condition is necessary.

Conversely, suppose that $\mu(x) \geq \mu(u)$ for all $x \in X$.

Now For $x \neq u, x \neq v$, $\min\{\mu^1(x \otimes b), \mu^1(b)\}$

$$= \min\{\mu(x), \mu(u)\} = \mu(u) \qquad \text{...(2.8)}$$

For $x=u, x \neq v$, $\min\{\mu^1(x \otimes b), \mu^1(b)\}$

$$= \min\{\mu(b), \mu(b)\} = \mu(u) \qquad \text{... (2.9)}$$

For $x=v, x \neq u$, $\min\{\mu^1(x \otimes b), \mu^1(b)\}$

$$= \min\{\mu(0), \mu(u)\} = \mu(u), \qquad \text{...(2.10)}$$

since $\mu(x) \geq \mu(u)$.

From (2.8), (2.9) and (2.10) we get

$$\mu^1(x) \geq \min\{\mu^1(x \otimes b), \mu^1(b)\}.$$

Also for $x \in X$ and $x \neq u, x \neq 0$,

$$\min\{\mu^1(b \otimes x), \mu^1(x)\} = \min\{\mu^1(b), \mu^1(x)\}$$

$$= \min\{\mu(u), \mu(x)\} = \mu(u) \quad \text{....(2.11)}$$

If $x = u$, $\min\{\mu^1(b \otimes u), \mu^1(u)\} = \min\{\mu(0), \mu(u)\} = \mu(u)$ ... (2.12)

If $x = 0$, $\min\{\mu^1(b \otimes 0), \mu^1(0)\} = \min\{\mu(u), \mu(0)\} = \mu(u)$ ... (2.13)

(since $\mu^1(b) = \mu(u)$).

So in all cases condition $\mu^1(b) \geq \min\{\mu^1(b \otimes x), \mu^1(x)\}$ is satisfied.

This proves that μ^1 is a fuzzy ideal on Y.

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