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CONTACT CR-PRODUCTS OF LORENTZIAN SASAKIAN MANIFOLD

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Contact CR-products have been studied by many authors Bejancu (1978), Blair and chen (1979), Yano and Mon (1981) and Kobayashi (1982) studied the contact CR-products of Sasakian manifolds.

In this paper we have studied the contact CR products of a LorentzianSasakian manifold Pablo Alegree (2011) and obtained in integrability conditions for the distributions D and $D^{\perp}$. Some properties of contact CR products of Lorentzian Sasakian manifolds have also been studied.

INTRODUCTION

An odd dimensional Riemannian manifold $(\widetilde{M}^{2n+1}, g)$ is called a Lorentzian almost contact manifold if it is endowed with a structure (ϕ, ξ, η, g), where ϕ is one – one tensor, ξ and η a vector field and one form on $\widetilde{M}$, respectively, and g is a Lorentz metric, satisfying

$$\phi^2 U = -U + \eta(U)\xi \quad \text{...(1.1)}$$

$$g(\phi U, \phi V) = g(U, V) + \eta(U)\eta(V) \quad \text{...(1.2)}$$

$$\eta(\xi) = 1 \quad \text{...(1.3)}$$

$$\eta(U) = -g(U, \xi) \quad \text{... (1.4)}$$

for any vector field U, V in $\widetilde{M}$. Let ϕ denote the 2-form in M given by $\phi(X, Y) = g(X, \phi Y)$if $d\eta = \phi$.

$\widetilde{M}$is called contact Lorentzian manifold. A normal contact Lorentzian manifold is called Lorentzian Sasakian T.I Kawa (1998), this is contact Lorentzian on satisfying.

$$(\widetilde{\nabla}_U \phi)V = -g(U, V) - \eta(V)U \quad \text{...(1.5)}$$

and $$\widetilde{\nabla}_U \xi = \phi U \quad \text{...(1.6)}$$

For any vector field U tangent to M, we put

$$\phi U = TU + FU \qquad \text{...(1.7)}$$

where TU (resp. FU) denotes the tangential (resp. normal) component of ϕU. Also for a normal vector field N, we put

$$\phi N = tN + fN \qquad \text{...(1.8)}$$

where tN (resp. fN) is the tangent part (resp. normal part) of ϕN.

Definition : A submanifold M is a *CR*-submanifold if M is tangent to ξ and there exists differential distributions $D: x \longrightarrow D_x \subset T_x M$ on M satisfying the following conditions,

$$\phi(D_x) \subset D_x \text{ for each } x \in M \qquad \text{...(1.9)}$$

and complementary orthogonal distribution

$$D^{\perp} : x \longrightarrow D^{\perp}{}_x \subset T_x M \qquad \text{...(1.10)}$$

is totally real *i.e.* $\phi D^{\perp}{}_x \subset T_x M^{\perp}$, for each $x \in M$. We call D (resp. $D^{\perp}$) horizontal (resp. vertical) distribution. The Gauss and Weingarten formulas are respectively given by

$$\tilde{\nabla}_U V = \nabla_U V + B(U, V) \qquad \text{...(1.11)}$$

$$\tilde{\nabla}_U N = -A_N U + \nabla_U N \qquad \text{...(1.12)}$$

where $\tilde{\nabla}$ is the Riemannian connection of M, ∇ is the Riemannian connection determined by the induced metric on M, ∇ is the connection in the normal bundle of M, B and A are both second fundamental tensor satisfying

$$B(UN, N) = g(A_N U, V) \qquad \text{...(1.13)}$$

Let $\widetilde{R}\,(R)$ be the curvature tensor of $\widetilde{M}$(resp. M), then the equation of Gauss and codazzi are respectively given by

$$g\big(\widetilde{R}\,(X,Y)Z, W\big) = g(R(X,Y)Z, W) - g\big(B(X,W), B(Y,Z)\big) \qquad \text{... (1.14)}$$

$$(\widetilde{R}\,(X,Y)Z)^{\perp} = \nabla_X B(Y,Z) - \nabla_Y B(X,Z) \qquad \text{...(1.15)}$$

BASIC LEMMAS AND PROPERTIES

Throughout this paper we have taken the structure tensor ξ as tangent to the submanifold. Since ξ is tangent to M for any vector field U tangent to M, we have

$$\tilde{\nabla}_U \xi = \phi U = \nabla_U \xi + B(U, \xi) \qquad \text{...(2.1)}$$

from which $\nabla_U \xi = TU$

and $$B(U, \xi) = FU \qquad \text{...(2.2)}$$

Lemma (2.1) Let *M* be a submanifold of a Lorentzian Sasakian manifold $(\widetilde{M},\phi,\xi,\eta,g)$. Then

$$(\nabla_U T)V = A_{FV}U + tB(U,V) - g(U,V)\xi - \eta(V)U \quad \text{... (2.3)}$$

$$(\nabla_U F)V = fB(U,V) - B(U,TV) \quad \text{... (2.4)}$$

for *U* and *V* be the vector field tangent to *M*.

Where we have defined $(\nabla_U T)V$ and $(\nabla_U F)V$ respectively by

$$(\nabla_U T)V = \nabla_U (TV) - T(\nabla_U V)$$

and $$(\nabla_U F)\, V = \nabla_U FV - F\nabla_U V$$

Proof : Differentiating (1.7) covariantly along *U* and using (1.7), (1.8) and separating the tangent and normal part, we get (2.3) and (2.4)

Lemma (2.2) Let *M* be a submanifold of a Lorentzian Sasakian manifold $(\widetilde{M},\phi,\xi,\eta,\theta)$. Then

$$(\nabla_U f)N = A_{fN}\, U - TA_N U \quad \text{...(2.5)}$$

$$(\nabla_U f)N = -FA_N U - B(U,tN) \quad \text{...(2.6)}$$

for any vector field *U* tangent to *M* and for any vector field *N* normal to M.

Where, we defined $(\nabla_U t)N$ and $(\nabla_U f)N$ respectively by

$$(\nabla_U t)N = \nabla_U (tN) - t\nabla_U N$$

and $$(\nabla_U f)N = \nabla_U fN - f\nabla_U{}^{\perp} N$$

Proof: Differentiating (1.8) covariantly along *U* and using (1.7), (1.8) and separating the tangent and normal parts we get (2.5) and (2.6)

Lemma (2.3) Let *M* be a submanifold of a Lorentzian Sasakian manifold $(\widetilde{M},\phi,\xi,\eta,g)$. Hence we have the following identities.

$$g(TU,V) = -g(U,TV), g(fN_1,N_2) = -g(N_1,fN_2) \quad \text{... (2.7)(a)}$$

$$g(FU,V) = -g(U,tV) \quad \text{...(2.7)(b)}$$

$$T\xi = 0, f\xi = 0 \quad \text{... (2.8)}$$

$$T^2U = -U + \eta(U)\xi - tFU \quad \text{...(2.9)}$$

$$FTU + fFU = 0 \quad \text{...(2.10)}$$

$$TtN + f^2N = -N \quad \text{...(2.11)}$$

$$FtN + tfN = 0 \quad \text{...(2.12)}$$

where *U* is a tangent vector field to *M* and *N*, N_1, N_2 are normal vector field to *M*.

Proof: Applying ϕ to (1.7) and (1.8) and separating the tangent and the normal parts we get the above identities.

Remark : Now we see that if *M* is tangent to ξ, then *M* is a CR-submanifold if and only if one of the following conditions is satisfied Kabayashi (1982)

(i) $FT = 0$ (ii) $tF = 0$

(iii) $tf = 0$ (iv) $Tt = 0$ ...(2.13)

Lemma 2.4. Let *M* be a submanifold of a Lorentzian Sasakian manifold $(\widetilde{M}, \phi, \xi, \eta, g)$. Then we have,

$$g(\nabla_U Z, X) = g(A_{FZ}U, TX) + \eta(\nabla_U Z)g(\xi, X) - \eta(Z)g(U, TX) \qquad ...(2.14)$$

for $X \in D$ and $Z \in D^{\perp}$

Proof : From (2.3), we see that for $Z \in D^{\perp}$

$$T\nabla_U Z = -A_{FZ}U - th(U, Z) + g(U, Z)\xi + \eta(Z)U \qquad ... (2.15)$$

For any $TX \in D$, we get

$$g(T\nabla_U Z, TX) = -g(A_{FZ}U, TX) - g(th(U, Z), TX) + g(U, Z)g(\xi, TX)$$
$$+g(U, TX)\eta(Z)$$

Which on using (2.9), gives

$$-g(\nabla_U Z, X) - g(tF\nabla_U Z, X) + \eta(\nabla_U Z)g(\xi, X) = -g(th(U, Z)$$
$$-g(A_{FZ}U, TX) + \eta(Z)g(U, TX) + g(U, Z)g(\xi, TX)$$

From which we get (2.14)

Lemma (2.5) : Let *M* be a submanifold of a Lorentzian Sasakian manifold $(\widetilde{M}, \phi, \xi, \eta, g)$. Then we have,

$$A_{FZ}W - A_{FW}Z = \eta(W)Z + \eta(Z)W \qquad ... (2.16)$$

for $Z, W \in D^{\perp}$

Proof: From (2.3), we see that $Z, W \in D^{\perp}$, we have

$$T[Z, W] = T\nabla_Z W - T\nabla_W Z$$
$$= -A_{FZ}U - th(U, Z) + g(U, Z) + \eta(Z)U + A_{FZ}W$$
$$+th(W, Z) - g(W, Z) - \eta(Z)W = 0$$

From which we get (2.16)

Proposition (2.1) : The horizontal distribution *D* of a CR-submanifold of a Lorentzian Sasakian manifold is integrable if and only if

$$g(h(X,TY),FZ) - g(h(Y,TX),FZ)$$
$$= \eta(Y)\nabla_X Z - \eta(X)\eta(\nabla_Y Z) + g(X,TY)\eta(Z) - g(Y,TX)\eta(Z)$$

for all $X, Y \in D$ *and* $Z \in D^{\perp}$ ...(2.17)

Proof: From (2.3), we see that for $Z \in D^{\perp}$

$$T\nabla_U Z = -A_{FZ}U - th(U,Z) + g(U,Z)\xi + \eta(Z)U$$

For any $TY \in D$, putting $U = X \in D$ and using (2.9), we get

$$g(Z,\nabla_X Y) = g(A_{FZ}X,TY) - \eta(Y)\eta(\nabla_X Z) - g(X,TY)\eta(Z) \quad \text{... (2.18)}$$

Similarly we can find,

$$g(Z,\nabla_Y X) = g(A_{FZ}Y,TX) - \eta(X)\eta(\nabla_Y Z) - g(Y,TX)\eta(Z) \quad \text{... (2.19)}$$

Subtracting equation (2.19) from equation (2.18), we get

$$g([X,Y],Z) = g(h(X,TY),FZ) - g(h(Y,TX),FZ) + \eta(X)\eta(\nabla_Y Z)$$
$$- \eta(Y)\nabla_X Z + g(Y,TX)\eta(Z) - g(X,TY)\eta(Z)$$

From which, we get (2.17)

Proposition (2.2) : If D is ξ – horizontal, then the leaf of $M^{\perp}$ of the vertical distribution $M^{\perp}$ of the vertical distribution $D^{\perp}$ is totally geoderic in M if and only if

$$g(h(Y,W),FZ) = g(W,Z)\eta(Y) \quad \text{... (2.20)}$$

for all $Y \in D$, and $W, Z \in D^{\perp}$

Proof : From (2.3), we see that for $Z \in D^{\perp}$

$$T\nabla_U Z = -A_{FZ}U - tB(U,Z) + g(U,Z)\xi + \eta(Z)U$$

Putting $U = W \in D^{\perp}$ and $Z \in D^{\perp}$ and using the fact that D is ξ-horizontal (resp. $D^{\perp}$), we get

$$T\nabla_W Z = -A_{FZ}W - tB(W,Z) + g(W,Z)\xi + \eta(Z)W$$

For any $Y \in D$, we get

$$g(T\,\nabla_W Z,Y) = -g(A_{FZ}W,Y) - g(tB\,(W,Z),Y) + g(W,Z)g(\xi,Y) + g(W,Y)\eta(Z)$$

Which on using (1.14), gives

$$g(\nabla_W Z,Y) = g(B(Y,W),FZ) - g(W,Z)\eta(Y)$$

From which we get (2.20)

3-Contact CR-Product

Definition: A CR-submanifold of a Lorentzian Sasakian manifold $\widetilde{M}$ is called a contact CR-product if it is locally Riemannian product of a Lorentzian Sasakian (invariant) submanifold $M^{\perp}$ and a totally real (anti-invariant) submanifold $M^{\perp}$ of $\widetilde{M}$ (Kobayashi 1982).

Theorem (3.1): Let *M* be a CR-submanifold of a LorentzianSasakian manifold. Then *T* is parallel if and only if *M* is anti-invariant submanifold.

Proof : From (2.3), We have

$$(\nabla_U T)V = A_{FV}U + th(U,V) - g(U,V)\xi - \eta(V)U$$

Putting $V = \xi$, using (1.3), (2.2), (2.8) and assure that *T* is parallel gives

$$0 = (\nabla_U T)\xi = -U + tFU - \eta(U)\xi \quad ...(3.1)$$

Operating *T* to equation (3.1) and using (2.8) and (2.13) gives

$$TU = 0$$

From which we see that M is anti-invariant submanifold.

Theorem (3.2) : Let M be a CR-submanifold and Let D be ξ – horizontal (resp. $D^{\perp}$). Then *M* is a contact CR-product if

$$A_{FZ}TX = 0 \text{ for } X \in D \text{ and } Z \in D^{\perp} \quad ... (3.2)$$

Proof : Since M is contact CR-product for any tangent vector X

$$\nabla_U X \in D \text{ for } X \in D \text{ and } \nabla_U Z \in D^{\perp} \text{ for } Z \in D^{\perp}$$

From (2.3), we see that for $Z \in D^{\perp}$

$$T\nabla_U Z = -A_{FZ}U - th(U,Z) + g(U,Z)\xi + \eta(Z)U \quad ... (3.3)$$

Now, from (3.3) and (2.9) and using the fact that D is ξ horizontanl.

$$0 = g(TA_{FZ}U,X) - g(tF\,\nabla_U Z,X)$$

Again

$$0 = g(\nabla_U Z,X) = -g(Z,\nabla_U X)$$

$$= -g(TA_{FZ}U,X) + g(tF\nabla_U Z,X)$$

$$= g(A_{FZ}TX\,,U) + g(tF\nabla_U Z,X)$$

Implying that (3.2) holds good.

Lemma (3.3) : Let M be a contact CR-product of a Lorentzian Sasakian manifold and D be ξ-horizontal (resp. $D^{\perp}$). Then for unit vector $X \in D$ with $\eta(X) = 0$ and $Z \in D^{\perp}$, we have

$$g(B(\nabla_X \phi X,Z),\phi Z) = -1 \quad ... (3.4)$$

$$g\big(B\big(\nabla_{\phi X} X, Z\big), \phi Z\big) = 1 \qquad \text{.... (3.5)}$$

$$g(B(\phi X, \nabla_X Z), \phi Z) = g\big(B\big(X, \nabla_{\phi X} Z\big), \phi Z\big) = 0 \qquad \text{... (3.6)}$$

Proof : By using (1.6) and $g(B(U,X), FZ) = \eta(X) g(Z,U)$, we can easily have

$$\begin{aligned} g(B(\nabla_X \phi X, Z), \phi Z) &= \eta\big(\nabla_{\phi X} X\big) g(Z,Z) \\ &= g(\nabla_X \phi X, \xi)\, g(Z,Z) \\ &= -g(\phi X, \nabla_X \xi) g(Z,Z) \\ &= -g(\phi X, \phi X)\, g(Z,Z) \\ &= -\big||X|\big|^2 ||Z||^2 \\ &= -1. \end{aligned}$$

From which we get (3.4). Similarly we can find (3.5) and (3.6)

Theorem (3.4) : Let *M* be a contact CR-product of a Lorentzian Sasakian manifold and *D* be ξ-horizontal (resp. $D^{\perp}$). Then for unit vector $X \in D$ with $\eta(X) = 0$ and $Z \in D^{\perp}$, we have

$$g\big(\tilde{R}(X, \phi X) X, \phi Z\big) = -2\, \big||B(X,Z)|\big|^2 + 2 \qquad \text{... (3.7)}$$

Proof : Using equation (1.16) and Lemma (3.3), we can easily have,

$$\begin{aligned} g\big(\tilde{R}(X, \phi X), \phi Z\big) &= g\Big((\nabla_X B)(\phi X, Z) - g\big(\nabla_{\phi X} B\big)(X,Z), \phi Z\Big) \\ &= g(D_X B(\phi X, Z), \phi Z) - g(D\phi_X B(X,Z), \phi Z) + 2 \\ &= g\big(B(X,Z), D_{\phi X} \phi Z\big) - g(B(\phi X, \mathrm{Z}), D_X \phi Z) + 2 \\ &= g\Big(B(X,Z), \phi\big(\tilde{\nabla}_{\phi X} Z\big)\Big) + g\big(B(X,Z), \big(\tilde{\nabla}_{\phi X} \phi\big) Z\big) + 2 \\ &\quad -g\Big(B(\phi X, Z), \phi\big(\tilde{\nabla}_X Z\big)\Big) - g\big(B(\phi X, Z), \big(\tilde{\nabla}_X \phi\big) Z\big) + 2 \\ &= -g\big(\phi B(X,Z), \phi B(X,Z)\big) - g\big(\phi B(X,Z)\,, \phi B(X,Z)\big) + 2 \\ &= -g\big(B(X,Z), B(X,Z)\big) - g\big(B(X,Z), (X,Z)\big) + 2 \\ &\qquad -2B\, \big||X,Z|\big|^2 + 2 \end{aligned}$$

From which we get (3.7) and the fact that *D* is ξ- horizontal.

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