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HYPER PARTIAL HOMOMORPHISMS IN BCH-ALGEBRAS

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The concept of partial homomorphism in BE-algebras has been studied by Pandey, Ilyas and Prasad **[2]** in 2018. Here we have developed a method to extend a BCK/BCI-algebra into a BCH-algebra. The concept of hyper partial homomorphism in a BCH-algebra has been considered with some results and suitable examples.

Keywords : BCK/BCI/BCH- algebras, partial and hyper partial homomorphism, disjoint elements.

PRELIMINARIES

Definition (1.1) :- Let $(X; *, 0)$ be a system where X is a non-empty set, '*' is a binary operation and '0' is a fixed element.

Then $(X; *,0)$ is called.

(a) a BCI - algebra **[3]** if the elements of X satisfy the following conditions:

(P1). $((x*y)*(x*z))*(z*y)=0$

(P2). $(x*(x*y))*y=0$

(P3). $x*x=0$

(P4). $x*y=0$ and $y*x=0$ imply $x=y$,

for all $x, y, z \in X$;

(b) a BCK-algebra **[3]** if the elements of a system $(X;*,0)$, in addition to above conditions, also satisfy:

(P5). $0 * x = 0$, *for all* $x \in X$.

Lemma (1.2) :- Elements of a BCK/BCI- algebra also satisfy the following conditions:

(P6). $x * 0 = x$,

(P7). $(x * y) * z = (x * z) * y$,

(P8). $x * (x * (x * y)) = x * y$,

(P9). $x \le y$ implies $x * z \le y * z$ and $z * y \le z * x$, for all $x, y, z \in X$,

where $x \le y$ means $x * y = 0$.

Definition (1.3) :- A system $(X;*,0)$ is called a BCH-algebra **[5]** if the elements of X satisfy only conditions (P3), (P4) and (P7).

Further, X is a positive BCH-algebra if elements of X also satisfy (P5).

Remark (1.4) :- The concept of a BCH-algebra is a generalization of the concepts BCK/BCI-algebra.

Definition (1.5) :-A subset M of a BCH-algebra $(X;*,0)$ is called a subalgebra if $x,\ y \in M \Rightarrow\ x * y \in M$. A subalgebra M is called a maximal subalgebra if it not contained in any subalgebra other than X .

Definition (1.6) :-A pair $\{x, y\}$ of distinct elements of X is said to be.

(a) mutually disjoint **[7]** if $x * y = x$ and $y * x = y$;

(b) semi-mutually disjoint if either $x * y = x$ and $y * x = 0$ or $x * y = 0$ and $y * x = y$

(c) co-equal if either $x * y = x$ and $y * x = x$ or $x * y = y$ and $y * x = y$.

Lemma (1.7) :- If a system $(X;*,0)$ with properties (P3), (P5), (P6) contains a pair $\{x, y\}$ of non-zero co-equal elements then (P1) can not be satisfied.

Proof :- Suppose that $x * y = x$ and $y * x = x$. Then $((y * x) * (y * 0)) * (0 * x)$ $= (x * y) * 0 = x \ne 0$ implies that (P1) is not satisfied.

We present here some extension theorems as follows :

Theorem (1.8) [9] :- Let $(X;*,0)$ be a BCH-algebra and let $t \notin X$.

Let $$Y = X \cup \{t\}.$$

We define a binary operation $\otimes$ $in Y\ as$

$$x \otimes y = x * y \ if \ x, y \in X, . \quad ...(1.1)$$

$$x \otimes t = x \ for\, x \neq 0;\ x \in X \ \text{ and } \ t \otimes x = t; x \in X \ . \quad ...(1.2)$$

$$0 \otimes t = t,\ t \otimes 0 = t,\ t \otimes t = 0 \quad ...(1.3)$$

Then $(Y; \otimes, 0)$ is a BCH-algebra iff X is a positive BCH-algebra.

Corollary (1.9) :- In case X is not a positive BCH-algebra then taking $0 \otimes t = 0$ in (1.3), $(Y, \otimes, 0)$ becomes a BCH-algebra.

Theorem (1.10) :-Every BCK/BCI-algebra with a pair of non-zero mutually disjoint elements can be extended to a BCH-algebra, by adjoining one element and defining a binary operation suitably, which is not a BCK/BCI-algebra.

Proof :- Let $(X; *, 0)$ be a BCK/BCI-algebra and let $u, v \in X$, be non-zero mutually disjoint elements of X. Let *b* be an element not in X and let $Y = X \cup \{b\}$. We define a binary operation $\otimes$ in *Y* as follows:

$$x \otimes y = x * y \ \text{ if } \ x, y \in X, \quad ...(1.4)$$

$$x \otimes b = x \ if \ x \neq u, x \neq v, x \in X, \quad ...(1.5)$$

$$u \otimes b = b \text{ and } v \otimes b = 0, \quad ...(1.6)$$

$$0 \otimes b = 0,\ b \otimes 0 = b,\ \ b \otimes b = 0, \quad ...(1.7)$$

$$b \otimes u = 0,\ b \otimes y = b \ \ for \ \ y \neq u. \quad ...(1.8)$$

For elements of $X \subseteq Y$, all conditions of a BCH-algebra are satisfied. Also from the given definitions of $\otimes$, (P3) and (P4) are satisfied. So it remains to check condition (P7) for elements, 0, *u*, *v* and *b*,

We have

$$(u \otimes v) \otimes b = u \otimes b = b, \qquad (u \otimes b) \otimes v = b \otimes v = b;$$

$$(b \otimes u) \otimes v = 0 \otimes v = 0, \qquad (b \otimes v) \otimes u = b \otimes u = 0;$$

$$(v \otimes b) \otimes u = 0 \otimes u = 0, \qquad (v \otimes u) \otimes b = v \otimes b = 0;$$

and

$$(0 \otimes u) \otimes b = 0 \otimes b = 0, \qquad (0 \otimes b) \otimes u = 0 \otimes u = 0;$$

If x and y are different from u and v then $(x \otimes y) \otimes b = t \otimes b = t$, where $x \otimes y = x * y = t$ (say) and $(x \otimes b) \otimes y = x \otimes y = t$.

Also $(b \otimes x) \otimes y = b \otimes y = b$ and $(b \otimes y) \otimes x = b \otimes x = b$.

So, $(Y, \otimes, 0)$ is a BCH-algebra.

Since $((v \otimes u) \otimes (v \otimes b)) \otimes (b \otimes u) = (v \otimes 0) \otimes 0 = v \neq 0,$

Y is not a BCK-algebra.

Remark (1.11) :-(i) In the above theorem pair $\{u, v\}$ may be taken as semi-mutually disjoint.

(ii) In the equation (1.7) we may take $0 \otimes b = b$.

HYPER PARTIAL HOMOMORPHISM

Definition (2.1) :-Let $(X;*,0)$ be a BCH-algebra and let $f : X \to X$ be a mapping.

(a) If $f(x*y) = f(x)*f(y)$ for all $x, y \in X$ then f is called a homomorphism.

(b) If there exists a subalgebra M of X such that $f(x*y) = f(x)*f(y)$ for all $x, y \in M$ but $f(x*y) \neq f(x)*f(y)$ for some $x, y \in X$ then f is called a partial homomorphism on X with respect to subalgebra M.

(c) If M is a maximal subalgebra of X then partial homomorphism defined on X with respect to M is called a hyper partial homomorphism.

Lemma (2.2):- If $f : X \to X$ is a homomorphism or partial homomorphism then $f(0) = 0$ and K_f is a subalgebra of X where $K_f = \{x \in X : f(x) = 0\}$.

Example (2.3) [9] :-Let $S = \{a, b, c, d, e\}$ and let $X = \{0, A, B, C, D, E, F, 1\}$ where $0 = \phi, A = \{a, b\}, B = \{a, b, c\}, C = \{c\}, D = \{c, d, e\}, E = \{d, e\}, F = \{a, b, d, e,\}, 1 = S.$

A binary operation 'o' is defined in X as follows :

For $L, K \in X$, $LoK = L \cap K^c$.

The binary operation table for '*o*' is given as.

Table (2.1)

o	0	A	B	C	D	E	F	1
0	0	0	0	0	0	0	0	0
A	A	0	0	A	A	A	0	0
B	B	C	0	A	A	B	C	0
C	C	C	0	0	0	C	C	0
D	D	D	C	E	0	C	C	0
E	E	E	E	E	0	0	0	0
F	F	E	E	F	A	A	0	0
1	1	D	E	F	A	B	C	0

The system $(X;o,0)$ is a BCH-algebra.

(a) Let M be the collection of those elements which do not contain a. For $K, L \in M$, $KoL = K \cap L^c$ does not contain a. So $KoL \in M$. This proves that M is a subalgebra of X .

We consider a mapping $f_a : X \to X$ as

$$f_a(L) = \begin{cases} L \ if \ a \notin L \\ L^c \ if \ a \in L. \end{cases}$$

Then for $L, K \in M$, $f_a(L) = L$ and $f_a(K) = K$.

Also $f_a(LoK) = f_a(L \cap K^c) = L \cap K^c$, $a \notin L \cap K^c$.

Futher, $f_a(L) \ o \ f_a(K) = LoK = L \cap K^c$.

This proves that $f_a(LoK) = f_a(L) \ o \ f_a(K)$ for all $L, M \in M$.

So, f_a is a homomorphism on sub algebra M.

Now $$f_a(A) = A^c = D, \ f_a(C) = C,$$

$$f_a(AoC) = f_a(A) = D$$

and $$f_a(A) o \ f_a(c) = DoC = E \neq D,$$

which means that $$f_a(A \, o \, C) \neq f_a(A) \ o \ f_a(c).$$

So f_a is a partial homomorphism on X with respect to subalgebra M such that $f_a(0) = 0$.

(b) Let $f, g : X \to X$ be defined as

$$f(x) = x \text{ for all } x \neq 1, f(1) = 0 \quad ...(2.2)$$

$$g(x) = 0 \text{ for all } x \neq 1, g(1) = 1 \quad ...(2.3)$$

Then f and g are homomorphisms on subalgebra

$$L = \{0, A, B, C, D, E, F\}.$$

But $f(C\,o\,1) = f(0) = 0$ and $f(C)\,o\,f(1) = C\,o\,0 = C$

$$g(1\,o\,E) = g(B) = 0 \text{ and } g(1)\,o\,g(E) = 1\,o\,0 = 1$$

imply that *f* and *g* are not homomorphisms on *X*.

Since *L* is a maximal subalgebra of *X*, f and g are hyper partial homomorphisms w.r.t. *L*.

SOME EXAMPLES.

We recall BCH-algebras X and Y discussed in theorem (1.8) and define:

Definition (3.1) :- For a given function f from X into itself, we define h_1, h_2, h_3 *on Y as*

$$h_1(x) = h_2(x) = h_3(x) = f(x); x \in X;$$

and $$h_1(t) = 0,\ h_2(t) = t \text{ and } h_3(t) = z \text{ for some fixed } z \neq 0, z \in X.$$

Now, we see that

Lemma (3.2) :- h_1 and h_2 are homomorphisms in Y iff f is a homomorphism in X.

Proof :- Suppose that h_1 and h_2 are homomorphisms on Y. Then

$$h_1(x) = h_2(x) = f(x); x \in X$$

implies that f is a homomorphism in X.

Suppose that f is a homomorphism in X. Then

$$f(0) = 0.$$ Now for $x \in X$, we have

$$h_1(x \otimes t) = h_1(x) = f(x),$$

$$h_1(x) \otimes h_1(t) = f(x) * 0 = f(x),$$

$$h_1(t \otimes x) = h_1(t) = 0$$

$$h_1(t) \otimes h_1(x) = 0 * f(x) = 0,$$

since X is positive.

Also $$h_1(0 \otimes t) = h_1(t) = 0$$

$$h_1(0) \otimes h_1(t) = f(0) \otimes 0 = 0,$$

This proves that h_1 is a homomorphism on Y .

Again, For $x \in X$, we have

$$h_2(x \otimes t) = h_2(x) = f(x),$$

$$h_2(x) \otimes h_2(t) = f(x) \otimes t = f(x);$$

$$h_2(t \otimes x) = h_2(t) = t,$$

$$h_2(t) \otimes h_2(x) = t \otimes f(x) = t.$$

Also, $$h_2(0 \otimes t) = h_2(t) = t,$$

$h_2(0) \otimes h_2(t) = 0 \otimes t = t$. So h_2 is a homomorphism on Y .

Lemma (3.3) :- h_3 is a hyper partial homomorphism on Y .

Proof :- Since $h_3(0 \otimes t) = h_3(t) = z$ and $h_3(0) \otimes h_3(t) = 0 * z = 0,$

we see that h_3 is not a homomorphism on Y .

Also $h_3(x) = f(x)$ is a homomorphism in X *and* X in a maximal subalgebra of Y . This proves that h_3 is a hyper partial homomorphism on Y .

We recall the BCH-algebras X and Y, elements u, v, b and binary operation $*$ and $\otimes$ discussed in the proof of theorem [1.10] and define:

Definition (3.4) :- Let f be a homomorphism on X . We consider functions g_1, g_2, g_3, g_4, g_5 on Y defined as

$$g_1(x) = g_2(x) = g_3(x) = g_4(x) = g_5(x) = f(x) \text{ for } x \in X$$

and $g_1(b)=0, g_2(b)=b, g_3(b)=u, g_4(b)=v$ and $g_5(b)=t$ for some fixed $t\in X$.

Lemma (3.5) :- g_1 is a homomorphism on Y iff $f(u)=0,$ *and* $f(v)=0.$

Proof :- Suppose that g_1 is a homomorphism on Y. Then $u\otimes b=b$ (Relation(1.6))

$$\Rightarrow \quad g_1(b)=g_1(u\otimes b)=g_1(u)\otimes g_1(b)$$

$$\Rightarrow \quad 0=f(u)*0 \Rightarrow f(u)=0 \qquad ...(3.1)$$

Again, $v\otimes b=0$

$$\Rightarrow \quad g_1(0)=g_1(v\otimes b)=g_1(v)\otimes g_1(b)$$

$$\Rightarrow \quad f(0)=f(v)*0$$

$$\Rightarrow \quad f(v)=0\,. \qquad (3.2)$$

Conversely, suppose that $f(u)=0$ and $f(v)=0$. Then g_1 is a homomorphism can be proved by simple calculation.

Remark (3.6) :- Above result shows that if f is a homomorphism such that either $f(u)\neq 0$ *or* $f(v)\neq 0$ then g_1 is not a homomorphism on Y, but it is a homomorphism w.r.t. *X*. So in this case g_1 is a hyper partial homomorphism on Y, because X is a maximal BCH-subalgebra of Y.

Lemma (3.7) :- g_2 is a homomorphism on Y iff conditions

(i) $f(u)=u$

(ii) $f(v)=v$

and (iii) $f: X-\{u,v\}\to X-\{u,v\}$ are satisfied.

Proof :- Let g_2 be a homomorphism on *Y*. Then $v\otimes b=0$

$$\Rightarrow \quad g_2(0)=g_2(v\otimes b)=g_2(v)\otimes g_2(b)$$

$$\Rightarrow \quad 0=f(v)\otimes b \qquad ...(3.3)$$

again, $b\otimes v=b$

$$\Rightarrow \quad b\otimes f(v)=b \qquad (3.4)$$

From the definition of binary operation $\otimes$ given in (1.4) to (1.8), (3.3) and (3.4) are possible only when $f(v) = v$.

again, $u \otimes b = b$

$\Rightarrow \quad g_2(u \otimes b) = g_2(b)$

$\Rightarrow \quad g_2(u) \otimes g_2(b) = b$

$\Rightarrow \quad f(u) \otimes b = b$ ...(3.5)

Also, $b \otimes u = 0$

$\Rightarrow \quad g_2(b \otimes u) = g_2(0) = f(0) = 0$

$\Rightarrow \quad g_2(b) \otimes g_2(u) = 0$

$\Rightarrow \quad b \otimes f(u) = 0$ ...(3.6)

From the definition of binary operation $\otimes$ given in theorem (1.10), (3.5) and (3.6) are possible only when $f(u) = u$.

Further,

$$x \otimes b = x \text{ and } b \otimes x = b \text{ for } x \neq u,\ x \neq v$$

$\Rightarrow \quad g_2(x \otimes b) = g_2(b)$ and $g_2(b \otimes x) = g_2(b)$

$\Rightarrow \quad f(x) \otimes b = b$ and $b \otimes f(x) = b$ ...(3.7)

The definition of binary operation $\otimes$ imply that (3.7) can the satisfied only when $f(x) \in X - \{u, v\}$,

i.e, $\quad f : (X - \{u, v\}) \to (X - \{u, v\})$

So conditions (i), (ii) and (iii) are satisfied.

Conversely, suppose that conditions (i), (ii) and (iii) are satisfied. Then simple computation shows that g_2 is a homomorphism.

Remark (3.8) :- From the above lemma it follows that if we take a homomorphism f not satisfying any one of the conditions of the above lemma, then g_2 is not a homomorphism on Y and in that case f is a hyper partial homomorphism on Y.

Lemma (3.9) :- g_3 is not a homomorphism on Y.

Proof :- If possible, suppose that g_3 is a homomorphism on Y, then the condition $b \otimes y = b$ for $y \neq u$.

$$g_3(b \otimes y) = g_3(b) = u$$

$\Rightarrow \quad g_3(b) \otimes g_3(y) = u$

$\Rightarrow \quad u * f(y) = u$ ...(3.8)

Again the condition, $y \otimes b = b$ *for* $y \neq u, y \neq v, y \in X$.

$$\Rightarrow \quad g_3(y \otimes b) = g_3(b) = u$$

$$\Rightarrow \quad g_3(y) \otimes g_3(b) = u$$

$$\Rightarrow \quad f(y) * u = u \qquad \text{... (3.9)}$$

Thus for $y \neq u$, $y \neq v$, condition (3.8) and (3.9) imply that

$$\big(\big(f(y)*u\big)*\big(f(y)*0\big)\big)*(0*u) = \big(u*f(y)\big)*0 = u*0 = u \neq 0.$$

This contradicts our assumption that X is a BCK/BCI-algebra. Hence g_3 is not a homomorphism.

Remark (3.10) :- From the above result it follows that for every homomorphism f defined on X, g_3 is not a homomorphism on Y. Since X is a maximal subalgebra of Y, g_3 is a hyper partial homomorphism on Y *w.r.t.* X. (Since every BCK/BCI-algebra is a BCH-algebra).

Remark (3.11) :-Similar arguments show that g_4 and g_5 are not homomorphisms on Y. So g_4 and g_5 are also hyper partial homomorphism on Y w.r.t. X .

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