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ON GENERALIZED (N, p_n, q_n) SUMMABILITY

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DEFINITIONS AND NOTATIONS

A series $\sum a_n$ is said to be strongly summable G (N, p, λ) with index σ $(\sigma \geq 1)$ to s, or simply summable $[G(N, p, \lambda]_\sigma$, if

$$\frac{1}{\left|\sum_{v=0}^{n} \in_{n-v} \mu_v\right|} \sum_{v=0}^{n} |B_v| \, |J_v^* - s|^\sigma = 0(1)$$

or

$$\frac{1}{|(\in * \mu)_n|} \sum_{v=0}^{n} |B_v||J_v^* - s|^\sigma = 0(1)$$

where $$(\in * \mu)_n = \sum_{v=0}^{n} \in_{n-v} \mu_v$$

we shall also use the notations

$$(\xi * \mu)_n = \sum_{v=0}^{n} \xi_{n-v} \mu_v$$

and $$(\psi * \mu)_n = \sum_{v=0}^{n} \psi_{n-v} \mu_v$$

We define the sequences of constants $\{c_n\}$, $\{k_n\}$and $\{l_n\}$ by means of the following identities [5]

$$c(z) = \sum_{n=0}^{\infty} c_n z^n = p\,(z)^{-1}, \quad c_{-1} = 0 \qquad \text{... (1.1)}$$

$$k(z) = \sum_{n=0}^{\infty} k_n z^n = q\,(z) p(z)^{-1}, \quad k_{-1} = 0 \qquad \text{... (1.2)}$$

$$l(z) = \sum_{n=0}^{\infty} l_n Z^n = p\,(z) q(z)^{-1}, \quad l_{-1} = 0 \qquad \text{... (1.3)}$$

INTRODUCTION

Generalizing the theorems of BORWEIN and CASS [2] for Nörlund summability of infinite series SINHA and KUMAR [6] and CHANDRA [3] proved the following theorems.

Theorem I : If $p_n > 0$ for all n $q_o > 0$, $q_n \geq 0$ for all $n > 0$, $\lambda \geq 1$ (N, q, α) is regular.

$$\sum_{n=0}^{\infty} a_n = 0[N, p, \alpha]_\lambda$$

and

$$\sum_{n=0}^{\infty} b_n \text{ is bounded } (N, q, \alpha),$$

then

$$\sum_{n=0}^{\infty} c_n = 0 \, (N, r, \alpha)$$

Theorem II: If $p_n > 0$ for all n, $q_o > 0$, $q_n \geq 0$ for all $n>0$, $\lambda \geqq 1$ (N, p, α) and (N, p, α) are regular.

$$\sum_{n=0}^{\infty} a_n = s \, [N, p, \alpha]$$

and

$$\sum_{n=0}^{\infty} b_n = t \, [N, q, \alpha]$$

then

$$\sum_{n=0}^{\infty} c_n = st \, [N, r, \alpha]$$

Theorem III: If $p_n > 0$, $q_n > 0$ and $\alpha_n > 0$ for all n, $\lambda \geqq 1$ (N, q, α) is regular,

$$\sum_{n=0}^{\infty} a_n = 0 \, [N, p, \alpha]_\lambda$$

and

$$\sum_{n=0}^{\infty} b_n \text{ is bounded } [N, q, \alpha]_\lambda, \text{ then}$$

$$\sum_{n=0}^{\infty} c_n = 0 \, [N, r, \alpha]_\lambda$$

3. The object of this paper is to establish these theorems for G(N, p, q) summability.

Theorem 1: If $p_n > 0$ for all n, $q_o > 0, q_n \geq 0$ for all n > 0, $\sigma \geqq 1$ $[G\,(N, q, \lambda)]$ is regular.

$$\sum_{n=0}^{\infty} a_n = 0 \, [\mathrm{G}(N, q, \alpha)]_\sigma$$

and $$\sum_{n=0}^{\infty} b_n \text{ is bounded } [G\ (N, q, \lambda)],$$

then $$\sum_{n=0}^{\infty} c_n = 0\ \ [G\ (N, r, \lambda)]$$

Theorem 2 : If $p_n > 0$ for all n $q_o > 0, q_n \geq 0$ for all n > 0, $\sigma \geqq 1$ [G (N, p, λ)] and [G (N, p, λ)] is regular.

$$\sum_{n=0}^{\infty} a_n = s[G\ (N, p, \lambda)]_\sigma$$

and $$\sum_{n=0}^{\infty} b_n = t[G\ (N, q, \lambda)]$$

then $$\sum_{n=0}^{\infty} c_n = st[G\ (N, r, \lambda)]$$

Theorem 3 : If $p_n > 0, q_n > 0$ and $\alpha_n > 0$ for all n, $\sigma \geqq 1$, [G (N, q, λ)] is regular.

$$\sum_{n=0}^{\infty} a_n = 0[\mathrm{g}(N, p, \lambda)]_\sigma$$

and $$\sum_{n=0}^{\infty} b_n \text{ is bounded } [\mathrm{g}(N, q, \lambda)]_\sigma$$

then $$\sum_{n=0}^{\infty} c_n = 0[\mathrm{g}(N, r, \lambda)]_\sigma$$

4. We shall use the following lemma for the proof of our theorem:

Lemma- Let $\lambda_n \neq 0$ for $n \geq 0$. Then the necessary and sufficient conditions that G (N, p, λ) should imply G (N, p, λ) are that

$$\sum_{i=0}^{n} \left| (\in * \mu)_i \sum_{v=i}^{n} \xi_{n-i}\, C_{b-i} \right| = 0\ ((\xi * \mu)_n) \qquad \text{... (4.1)}$$

and

$$k_{n-i} = \sum_{v=i}^{n} \xi_{n-i}\ C_{v-i} = 0\ ((\xi * \mu)_n) \qquad \text{... (4.2)}$$

PROOF OF THE LEMMA

Writing

$$J_n^{(\alpha)} = (\in * \mu)^{-1} \sum_{v=0}^{n} \in_{n-v} t_v^{\alpha}\ \mu_v$$

we have by a similar inversion formula

$$t_n^\alpha = \mu_n^{-1} \sum_{v=0}^{n} c_{n-v} \ (\in * \mu)_v \, J_v^{(\alpha)}$$

thus

$$\zeta_n^{(\alpha)} = (\xi * \mu)_n^{-1} \sum_{v=0}^{n} \xi_{n-v} \ t_v^\alpha \ \mu_v$$

$$= (\xi * \mu)_n^{-1} \sum_{v=0}^{n} \xi_{n-v} \sum_{i=0}^{v} c_{v-i} \times (\xi * \mu)_i \ J_i^{(\alpha)}$$

where

$$d_{n,i} = \begin{cases} \dfrac{(\xi * \mu)_i}{(\xi * \mu)_n} \displaystyle\sum_{v=1}^{n} \xi_{n-v} \ c_{v-i} & (i \le n) \\ 0 & (i > n) \end{cases} \qquad \text{... (4.3)}$$

If, $s_n = 1$ for all n, then $J_n^{(\alpha)} = 1$ and $\zeta_n^{(\alpha)} = 1$, so that

$$\sum_{i=0}^{n} d_{n,i} = 1 \text{ for every } n. \qquad \text{... (4.4)}$$

Hence it follows from TOEPLITZ'S theorem [5] (Th. 2) that the transformation (4.3) is regular if and only if conditions (4.1) and (4.2) of the lemma are satisfied.

For $\alpha = 1$, this lemma reduces to the DAS's lemma $[[4], lemma\ 1]$

PROOF OF THE THEOREM 1 :

Using Hölder's inequality, it is easy to show that $\sigma > 1$,

$$[G(N, p, \lambda)]_\sigma \ \Rightarrow \ [G(N, p, \lambda)]_1 \qquad [1]$$

Thus it is sufficient of prove theorem 1 for case $\sigma = 1$.

Let

$$\omega_n^* = \frac{1}{(\in * \mu)_n} \sum_{v=0}^{n} (\xi * \mu)_{n-v} \ b_v$$

and

$$v_n^* = \frac{1}{(\psi * \mu)_n} \sum_{v=0}^{n} (\psi * \mu)_{n-v} \ c_v$$

Now $(\psi * \mu)_n \ v_n^*$ is the coefficient of x_n in the series.

$$\sum_{n=0}^{\infty} (\Delta \in * \mu)_n \, J_v^* \times^n \quad \sum_{n=0}^{\infty} (\xi * \mu)_n \ \omega_n^* \ x^n$$

$$= (\Delta \in * \mu)\ (x) a\ (x) (1 - x)^{-1} \ (\Delta \xi * \mu)\ x b(x)$$

$$= (\Delta \in * \mu)\ (x)\ (\Delta \xi * \mu)\ (x) (1 - x)^{-1} \ c(x)$$

$$= (\psi * \mu)\ (x) c(x)$$

Thus

$$(\psi * \mu)_n \;\; v_n^* = \sum_{v=0}^{\infty} (\Delta \in * \, \mu)_v \; j_v^* \; (\xi * \mu)_v \; |\omega_{n-v}^*|$$

and so
$$(\psi * \mu)_n \; | \, v_n^*| \leqq \sum_{v=0}^{\infty} (\Delta \in * \, \mu)_v \; |j_v^*| \; (\xi * \mu)_{n-v} \; |\omega_{n-v}^*|$$

since by hypothesis

$$\sum_{r=0}^{\infty} (\Delta \in * \, \mu)_r \; |j_v^*| = 0 (\in * \, \mu)_n$$

and
$$|\omega_n^*| \;\; = 0 \; (1)$$

it follow that

$$(\psi * \mu)_n \; | \, v_n^*| \;\; \leqq H \sum_{v=0}^{\infty} (\Delta \in * \, \mu)_v \; |j_v^*| \;\; (\xi * \mu)_{n-v}$$

$$= \; H \sum_{v=0}^{\infty} (\Delta \xi * \mu)_{n-v} \; \sum_{r=0}^{\infty} (\Delta \in * \, \mu)_r \; |j_v^*|$$

$$= \; H \sum_{v=0}^{\infty} (\Delta \xi * \mu)_{n-v} \; 0 \;\; (\in * \, \mu)_r$$

Since [G (N, q, λ)] is regular, the final sum is 0 $(\psi * \mu)_n$.

Thus

$$v_n^* \;\; = 0 \; (1), \;\; and \; so$$

$$\sum_{n=0}^{\infty} c_n \; = 0 \; [G(N, r, \lambda)] \text{ as required}$$

PROOF OF THE THEOREM 2 :

If $s = 0$ the result is an immediate consequence of theorem 1. Suppose $s \neq 0.$ Let

$$a_o' \;=\; a_o - s, \;\; a_n' \;=\; a_n \qquad for \;\; n > 0$$

and

$$c_n' \;\; = \; \sum_{v=0}^{n} a_v' \;\; b_{n-v}$$

then
$$c_n' \;\; = \;\; c_n \; - \;\; sb_n$$

Thus, since

$$\sum_{n=0}^{\infty} a_n' = 0 \; [G(N, p, \lambda)]_1$$

by hypothesis, we have

$$\sum_{n=0}^{\infty} c_n^{'} = 0\ [G(N, r, \alpha)]$$

by theorem 1. Further since $[G(N, p, \lambda)]$ is regular

$$[G(N, q, \lambda)] \ \Rightarrow [G(N, r, \lambda)]$$

Thus
$$\sum_{n=0}^{\infty} b_n = t\ [G(N, r, \lambda)]$$

Hence since

$$\sum_{n=0}^{m} c_n = \sum_{n=0}^{m} c_n^{'} + s \sum_{n=0}^{m} b_n \quad \text{for } m = 0, 1, 2, \dots \dots \dots$$

it follows that

$$\sum_{n=0}^{\infty} c_n = st\ [G(N, r, \lambda)]$$

This completes the proof.

PROOF OF THE THEOREM 2 :

Let

$$v_n^* = \frac{(f * \mu c)_n}{(\Delta\psi * \mu)_n}$$

and

$$\omega_n^* = \frac{(l * \mu b)_n}{(\Delta\xi * \mu)_n}$$

Now

$$\sum_{v=0}^{\infty} (\Delta\psi * \mu)_n\, v_n^*\, x_n$$

$$= \sum_{v=0}^{\infty} (\Delta\in * \mu)_n\, J_n^*\, x^n \quad \sum_{n=0}^{\infty} (\Delta\xi * \mu)_n\, \omega_n^*\, x_n$$

Thus
$$(\Delta\psi * \mu)_n\, v_n^* = \sum_{v=0}^{\infty} (\Delta\in * \mu)_v\, J_v^*\, (\Delta\xi * \mu)_{n-v}\, \omega_{n-v}^*$$

and so we have

$$\{(\Delta\psi * \mu)_n\, |v_n^*|\}^\sigma \ \leqq \ \left\{ \sum_{v=0}^{\infty} (\Delta\in * \mu)_v\, |j_v^*|\ (\Delta\xi * \mu)_{n-v} |\omega_{n-v}^*| \right\}^\sigma$$

Using Holder’s inequality, we find that

$$\{(\Delta\psi * \mu)_n\ |v_n^*|\}^\sigma \leqq \left\{\sum_{v=0}^{\infty}(\Delta\in * \mu)_v\ |j_v^*|^\sigma\ (\Delta\xi * \mu)_{n-v}|\omega_{n-v}^*|^\sigma\right\}$$

$$\times\left\{\sum_{v=0}^{\infty}(\Delta\in * \mu)_v\ (\Delta\xi * \mu)_{n-v}\right\}^{\sigma-1}$$

Thus

$$\sum_{n=0}^{\infty}(\Delta\psi * \mu)_n\ |v_n^*|^\sigma \leqq \sum_{n=0}^{m}\sum_{v=0}^{n}(\Delta\in * \mu)_v\ |j_v^*|^\sigma\ (\Delta\xi * \mu)_{n-v}|\omega_{n-v}^*|^\sigma$$

$$= \sum_{v=0}^{\infty}(\Delta\in * \mu)_v\ |j_v^*|^\sigma\ \sum_{v=i}^{m}(\Delta\xi * \mu)_{n-v}|\omega_{n-v}^*|^\sigma$$

Now, by hypothesis

$$\sum_{v=0}^{m}(\Delta\in * \mu)_v\ |j_v^*|^\sigma = 0\ (\in * \mu)_m$$

and

$$\sum_{v=0}^{m}(\Delta\xi * \mu)_v\ |\omega_v^*|^\sigma = 0\ (\xi * \mu)_m$$

thus

$$\sum_{v=0}^{m}(\Delta\psi * \mu)_n\ |v_n^*|^\sigma \leqq H\ \sum_{v=0}^{m}(\Delta\in * \mu)_v\ |j_v^*|^\sigma\ (\xi * \mu)_{m-v}$$

$$= H\ \sum_{v=0}^{m}(\Delta\xi * \mu)_{m-v}\ \sum_{\sigma=0}^{v}(\Delta\in * \mu)_r\ |j_r^*|^\sigma$$

$$= H\ \sum_{v=0}^{m}(\Delta\xi * \mu)_{m-v}\ o\ (\Delta\in * \mu)_v$$

$$= 0\ (\psi * \mu)_m$$

Since $[G\ (N, q, \lambda]$ is regular. Thus

$$\sum_{n=O}^{\infty} C_n = 0\ [G(N, r, \lambda)]_\sigma \text{ as required.}$$

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