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AN ESTIMATE OF ULTRASPHERICAL SERIES BY GENERALIZED NÖRLUND MEANS

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DEFINITIONS AND NOTATIONS

(i) Let $f(\theta,\phi)$ be a function defined for the range $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$ on a sphere S. We suppose throughout that the function

$$f(\theta', \emptyset')[\sin^2\theta' \sin^2(\emptyset - \emptyset')]^{\lambda-\frac{1}{2}} \qquad ...(1.1)$$

is absolutely integral (L) over the whole surface of the unit sphere.

A generalized mean value of $f(\theta,\emptyset)$ on the sphere has been defined by KOGBETLIANTZ [4, 5, 6, 7 and 8], we define the generalized mean value of $f(\theta,\emptyset)$ as follows

$$f(\omega) = \frac{\left\lceil\left(\frac{1}{2}\right)\right\lceil\left(\frac{1}{2}+\lambda\right)}{(\lambda)\ \ 2\pi\ (\sin\omega)^{2\lambda}} \int_{c_\omega} \frac{f(\theta', \emptyset')}{[\sin^2\theta' \sin^2(\emptyset - \emptyset')]^{\frac{1}{2}-\lambda}} \qquad ...(1.2)$$

where the integral is taken along the small circle whose centre is $(\theta,\emptyset)$ on the sphere and whose curvilinear radius is ω.

(ii) (N, p, q) means of ultraspherical series – It is known that SZEGO [10]

$$\sum (k+\lambda)\, P_n^{(\lambda)}(\cos\theta) = \frac{1}{2}\frac{(m+2\lambda)P_m^{(\lambda)}(\cos\theta) - P_{m+1}^{(\lambda)}(\cos\theta)(m+1)}{1-\cos\emptyset} \qquad ...(1.3)$$

$$= \frac{1}{2}\left[\frac{d}{dx}\left\{P_m^{(\lambda)}(x) + P_{m+1}^{(\lambda)}(x)\right\}_{x=\cos\theta}\right]$$

So the *m*th partial sum S_m of the series is given by

$$S_m = \frac{\lceil(\lambda)}{2\left\lceil\left(\frac{1}{2}\right)\right\lceil\left(\frac{1}{2}+\lambda\right)} \int_0^\pi f(\omega)\left[\frac{d}{dx}\{p_{m+1}^{(\lambda)}(x) + P_m^{(\lambda)}(x)\}\right]_{x=\cos\theta} (\sin\omega)^{2\lambda}\, d\omega$$

Now using the orthogonal property of the ultraspherical polynomials we have

$$S_m - f(P) = \frac{\lceil(\lambda)}{2\left\lceil\left(\frac{1}{2}\right)\right\lceil\left(\frac{1}{2}+\lambda\right)} \int_0^\pi F(\omega)\left[\frac{d}{dx}\{p_{m+1}^{(\lambda)}(x) + P_m^{(\lambda)}(x)\}\right]_{x=\cos\theta} (\sin\omega)^{2\lambda}\, d\omega$$

…(1.4)

where $f(P)$ is the value of the function at a point P on the sphere.

Putting

$$F(\omega) = [f(\omega) - f(P)] (\sin \omega)^{2\lambda-1} \quad \text{... (1.5)}$$

Hence, in virtue of the definition of (N, p, q) means, we have

$$t_n^{p,q} - f(P) = \int_0^{\pi} F(\omega)\ L_n(\omega) d\omega \quad \text{... (1.6)}$$

where $$L_n(\omega) = \frac{\lceil(\lambda)}{2\lceil\left(\frac{1}{2}\right)\lceil\left(\frac{1}{2}+\lambda\right)} \frac{1}{R_n} \sum_{k=0}^{\infty} p_{n-k} q_k \left[\frac{d}{dx}\{p_m^{(\lambda)}(x) + P_{m+1}^{(\lambda)}(x)\}\right]_{x=\cos\theta} \sin \omega$$

where $\{p_n\}$ and $\{q_n\}$ are positive and $\{q_n\}$ is non increasing sequences of real number such that

$$R_n = (p * q)_n = p_0 q_n + p_1 q_{n-1} + \cdots \ldots \ldots .. + p_0 q_0 (\neq 0)$$

$$p_{-1} = q_{-1} = r_{-1} = 0$$

We suppose throughout that

$$R_n{}^{\delta-1} \geq n^{\lambda-1}, n = 1, 2, \ldots \ldots$$

and $$\int_0^t R^{\delta}_{\left(\frac{1}{u}\right)}\ du = 0 \left[R^{\delta}_{\left(\frac{1}{t}\right)}\right]$$

INTRODUCTION

Generalizing the theorem of PORWAL [9], GUPTA and PANDEY [3] have proved a theorem on the degree of approximation to a function f(x) by Nörlund means of Fourier series.

Later on, generalizing the theorem of GUPTA and PANDEY [3] BEOHAR [1] has proved a theorem on the degree of approximation of a function by Nörlund means of ultraspherical series in the following form.

Theorem : Let $\{p_n\}$ be a positive non increasing sequence of real numbers such that $\left\{\frac{(P_n)^{\delta}}{n^{\lambda}}\right\}$ is increasing and

$$\int_t^{\delta} \frac{\frac{|F(\emptyset)|P}{\left(\frac{1}{\emptyset}\right)} d\emptyset}{\emptyset^{\lambda+1}} = 0 \left[t^{\lambda} \left(P_{\left(\frac{1}{t}\right)}\right)^{\delta}\right] \text{ for } 0 < \lambda < 1, 0 < \delta < 1 \quad \text{... (2.1)}$$

then $$t_n - f(P) = 0 \left(\frac{1}{P_n}\right)^{1-\delta} \quad \text{... (2.2)}$$

It may be mentioned here that the condition (2.1) is un-natural, because the right hand side approaches to zero $t \to 0$.

3. The object of the present paper is to improve the theorem on the degree of approximation to a function by its (N, p, q) means of ultraspherical series. However, our theorem is as follows.

Theorem : If $0 < \delta \leq \pi$, $0<\lambda<1$

$$\int_t^\delta \frac{|F(\omega)|P\left(\frac{1}{\omega}\right)d\omega}{\omega^{\lambda+1}} = 0\left[\left(R_{\left(\frac{1}{t}\right)}\right)^\delta\right] \quad \text{as } t \to 0 \qquad \text{... (3.1)}$$

and then

$$t_n - f(R) = 0 \left[\frac{n^{\lambda-1}}{(R_n)^{1-\delta}}\right] + 0\left[\frac{n^\lambda}{R_n}\right] \qquad \text{... (3.2)}$$

4. For the proof of the theorem we require the following lemmas

Lemma 1: [10] If $0 < \lambda < 1$ and C is a fixed constant and $n \to \infty$, then

$$P_n^{(\lambda)}(\cos\theta) = \begin{Bmatrix} \theta^{-\lambda}0\left(n^{\lambda-1}\right), & \frac{c}{n} \leq \theta \leq \frac{\pi}{2} \\ 0\left(n^{2\lambda-1}\right), & 0 \leq \theta \leq \frac{c}{n} \end{Bmatrix} \qquad \text{... (4.1)}$$

Lemma 2: [2] If $0 < \lambda < 1$ and $0 \leq \omega \leq \pi$, then

$$L_n(\omega) = 0\left(n^{2\lambda+1}\omega\right) \qquad \text{... (4.2)}$$

Lemma 3: [2] If $0 < \lambda < 1$ and $\pi - \frac{c}{n} \leq \omega \leq \pi$, then

$$L_n(\omega) = 0\left(n^{2\lambda}\sin\omega\right) \qquad \text{... (4.3)}$$

where c is a positive constant

Lemma 4: The condition (3.1) implies that

$$\int_o^t |F(\omega)|d\omega = 0\left[\frac{t^{\lambda+1}}{\left(R_{\left(\frac{1}{t}\right)}\right)^{1-\delta}}\right] \qquad \text{... (4.4)}$$

Proof of the lemma we write

$$\phi(t) = \int_t^\delta \frac{|F(\omega)|R_{\left(\frac{1}{\omega}\right)}d\omega}{\omega^{\lambda+1}} = 0\left[R_{\left(\frac{1}{t}\right)}^{\ \delta}\right]$$

Hence, on integration by parts, we get

$$\int_t^\delta |F(\omega)|R_{\left(\frac{1}{\omega}\right)}d\omega = \int_t^\delta u^{\lambda+1}\phi'(u)\,du$$

$$= \left[u^{\lambda+1}\emptyset(u)\right]_o^t - \int_o^t u^\lambda \emptyset(u)du$$

$$= 0\left[t^{\lambda+1}\left(R_{\left(\frac{1}{t}\right)}\right)^\delta\right] + 0\int_o^t u^\lambda \left(R_{\left(\frac{1}{u}\right)}\right)^\delta du$$

$$= 0\left[t^{\lambda+1}\left(R_{\left(\frac{1}{t}\right)}\right)^\delta\right]$$

Hence,

$$\int_0^t | F(\omega) | R_{\left(\frac{1}{\omega}\right)} d\omega = 0 \left[t^{\lambda+1} \left(R_{\left(\frac{1}{t}\right)} \right)^{\delta} \right]$$

$$R_{\left(\frac{1}{t}\right)} \int_0^t |F(\omega)| d\omega = 0 \left[t^{\lambda+1} \left(R_{\left(\frac{1}{t}\right)} \right)^{\delta} \right]$$

$$\int_0^t |F(\omega)| d\omega = 0 \left[\frac{t^{\lambda+1}}{\left(R_{\left(\frac{1}{t}\right)} \right)^{1-\delta}} \right]$$

Thus the lemma holds.

PROOF OF THE THEOREM

We have from (1.6)

$$t_n{}^{p,q} - f(P) = \int_o^{\pi} |F(\omega)| L_n(\omega)\, d\omega$$

$$= \int_o^{c/n} + \int_{c/n}^{\delta} + \int_{\delta}^{\pi - c/n} + \int_{\pi - \frac{c}{n}}^{\pi} |F(\omega)| L_n(\omega)\, d\omega$$

$$= I_1 + I_2 + I_3 + I_4 \qquad say \qquad \dots (5.1)$$

We first consider,

$$I_1 = \int_0^{\frac{c}{n}} |F(\omega)| L_n(\omega)\, d\omega$$

$$= 0 \left(n^{2\lambda+1} \right) \int_0^{\frac{c}{n}} |F(\omega)| \omega\, d\omega$$

Integrating by parts and using lemma 4, we get

$$I_1 = 0 \left[n^{2\lambda+1} \left(\frac{\omega^{\lambda+2}}{R_{\left(\frac{1}{\omega}\right)}{}^{1-\delta}} \right)_0^{c/n} \right]$$

$$= 0 \left[\frac{n^{2\lambda+1}}{(R_n)^{1-\delta}} . n^{-\lambda-2} \right]$$

$$= 0 \left[\frac{n^{\lambda-1}}{(R_n)^{1-\delta}} \right] \qquad \dots (5.2)$$

Next we consider I_2

$$I_2 = \int_{c/n}^{\delta} |F(\omega)| L_n(\omega)\, d\omega$$

$$= 0\left(\frac{n^{\lambda-1}}{R_n}\right)\int_{c/n}^{\delta}|F(\omega)|R_{\left(\frac{1}{\omega}\right)}\left(\sin\frac{\omega}{2}\right)^{-\lambda-1}\left(\cos\frac{\omega}{2}\right)^{-\lambda}d\omega$$

$$+0\left(\frac{n^{\lambda}}{R_n}\right)\int_{c/n}^{\delta}|F(\omega)|R_{\left(\frac{1}{\omega}\right)}\left(\sin\frac{\omega}{2}\right)^{-\lambda}\left(\cos\frac{\omega}{2}\right)^{1-\lambda}d\omega$$

$$= I_{2.1} + I_{2.2} \qquad say$$

We discuss $I_{2.1}$, first,

$$I_{2.1} = 0\left(\frac{n^{\lambda-1}}{R_n}\right)\int_{c/n}^{\delta}\frac{|F(\omega)|R_{\left(\frac{1}{\omega}\right)}\,d\omega}{\omega^{\lambda+1}}$$

$$= 0\left(\frac{n^{\lambda-1}}{R_n}\right)\left((R_n)^{\delta}\right)$$

$$= 0\left[\frac{n^{\lambda-1}}{(R_n)^{1-\delta}}\right] \qquad \dots (5.3)$$

Next we have

$$I_{2.2} = 0\left(\frac{n^{\lambda}}{R_n}\right)\int_{\frac{c}{n}}^{\delta}\frac{|F(\omega)|R_{\left(\frac{1}{\omega}\right)}\,d\omega}{\omega^{\lambda}}$$

$$= 0\left(\frac{n^{\lambda}}{R_n}\right)\int_{\frac{c}{n}}^{\delta}\frac{\omega|F(\omega)|R_{\left(\frac{1}{\omega}\right)}\,d\omega}{\omega^{\lambda+1}}$$

$$= 0\left(\frac{n^{\lambda}}{R_n}\right)\left[\frac{1}{n}\int_{\frac{c}{n}}^{\delta}\frac{|F(\omega)|R_{\left(\frac{1}{\omega}\right)}\,d\omega}{\omega^{\lambda+1}}\right]$$

$$= 0\left[\frac{n^{\lambda-1}}{R_n}\right]\left[(R_n)^{\delta}\right]$$

$$= 0\left[\frac{n^{\lambda-1}}{(R_n)^{1-\delta}}\right] \qquad \dots (5.4)$$

Combining (5.3) and (5.4), we get

$$I_2 = 0\left[\frac{n^{\lambda-1}}{(R_n)^{1-\delta}}\right] \qquad \dots (5.5)$$

Now, we consider I_3,

$$I_3 = \int_{\delta}^{\pi-\frac{c}{n}}|F(\omega)|L_n(\omega)d\omega$$

$$= 0\left(\frac{n^{\lambda-1}}{R_n}\right)\int_{\delta}^{\pi-\frac{C}{n}}\frac{|F(\omega)|R_{\left(\frac{1}{\omega}\right)}\,d\omega}{\left(\sin\frac{\omega}{2}\right)^{\lambda+1}\left(\cos\frac{\omega}{2}\right)^{\lambda}}$$

$$+ 0\left(\frac{n^{\lambda}}{R_n}\right)\int_{\delta}^{\pi-\frac{C}{n}}\frac{|F(\omega)|R_{\left(\frac{1}{\omega}\right)}\,d\omega}{\left(\sin\frac{\omega}{2}\right)^{\lambda}\left(\cos\frac{\omega}{2}\right)^{\lambda-1}}$$

$$= 0 \left[\frac{n^{\lambda-1}}{(R_n)^{1-\delta}}\right] + 0 \left[\frac{n^{\lambda}}{R_n}\right] \qquad \ldots (5.6)$$

At last, we consider I_4,

$$I_4 = \int_{\pi-\frac{c}{n}}^{\pi} |F(\omega)| L_n(\omega) d\omega$$

By lemma 3, we have

$$I_4 = 0 \left[n^{2\lambda} \int_{\pi-\frac{c}{n}}^{\pi} |F(\omega)| \sin\omega \; d\omega\right]$$

Putting $\omega = \pi - t,$ we obtain

$$I_4 = 0 \left[n^{2\lambda} \int_{o}^{\frac{c}{n}} t dt\right]$$

$$= 0 \left[n^{2\lambda-2}\right] \qquad \ldots (5.7)$$

Combining (5.2), (5.5), (5.6) and (5.7), we get

$$t_n - f(R) = 0 \left[\frac{n^{\lambda-1}}{(R_n)^{1-\delta}}\right] + 0 \left[\frac{n^{\lambda}}{R_n}\right]$$

Hence the theorem holds

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