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ALMOST INCREASING SEQUENCES AND THEIR APPLICATIONS

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DEFINITIONS AND NOTATIONS:

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with $\{S_n\}$ as the sequence of its partial sums. Let (σ_n) and (t_n) denote the n-th $(C, 1)$ means of the sequence $\{S_n\}$ and $\{na_n\}$ respectively. The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|C,1|_k, k \geq 1$, if [4]

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n - \sigma_{n-1}|_k < \infty \qquad \ldots (1.1)$$

In view of the fact that $t_n = n(\sigma_n - \sigma_{n-1})[6]$, equation (1.1) can be written as

$$\sum_{n=1}^{\infty} \frac{|t_n|^k}{n} < \infty \qquad \ldots (1.2)$$

Let $\{p_n\}$ be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty (p_{-i} = p_{-i} = 0, i \geq 1) \qquad \ldots (1.3)$$

The sequence-to-sequence transformation

$$\omega_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \, s_v \qquad \ldots (1.4)$$

defines the sequence $\{\omega_n\}$ of the $(\bar{N}, p_n)$ means of the sequence $\{S_n\}$ generated by the sequence of coefficients (p_n) [5]. The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $(\bar{N}, p_n)_k$, $k \geq 1$, if [2]

$$\sum_{v=0}^{n} \left(\frac{p_n}{P_n} \right)^{k-1} |\omega_n - \omega_{n-1}|^k < \infty \qquad \ldots (1.5)$$

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Let $\{\phi_n\}$ be any sequence of positive real constants. Then the series $\sum_{k=0}^{\infty} a_n$ is said to be summable $|\bar{N}, p_n, \phi_n, \delta, \beta|_k$ $k \geq 1$, $\delta \geq 0$ and $\beta \geq 1$, if

$$\sum_{n=1}^{n} \phi_n \, \beta^{(\delta k+k-1)} \, |\, \omega_n - \omega_{n-1} \,|^k < \infty \qquad \ldots(1.6)$$

For $\delta = 0$ and $\beta = 1$, our definition reduces to (1.5) [2]

We need the concept of almost increasing sequence. A positive sequence $\{b_n\}$ is said to be almost increasing if there exists a positive increasing sequence $\{c_n\}$ and two positive constant A and B such that

$$Ac_n \leq b_n \leq Bc_n \qquad [1]$$

Obviously every increasing sequence is almost increasing sequence but the converse need not be true as can be seen from the example $b_n = n \ \exp \ (-1)^n$

INTRODUCTION

Generalizing the theorem of BOR [3] for $|\bar{N}, p_n|_k$ summability factors of an infinite series TRIPATHI and PATEL [8] proved the following theorem for $|\bar{N}, p_n, \phi_n|_k$ summability.

Theorem : Let $\{X_n\}$ be a almost increasing sequence and the sequences $\{\lambda_n\}$ and $\{p_n\}$ are such that

$$P_n = 0(np_n) \text{ as } n \to \infty \qquad \ldots(2.1)$$

$$\lambda_m X_m = 0(1) \text{ as } m \to \infty \qquad \ldots(2.2)$$

$$\sum_{n=1}^{m} nX_n \,|\, \Delta^2 \lambda_n \,| = 0(1) \qquad \ldots(2.3)$$

and

$$\sum_{n=1}^{m} \phi_n^{k-1} \left(\frac{p_n}{P_n} \right)^k |\, t_n \,|^k = 0(X_m) \text{ as } m \to \infty \qquad \ldots(2.4)$$

where $\{\phi_n\}$ be a sequence of positive real constants such that $\left\{ \frac{\phi_n p_n}{P_n} \right\}$ is non-increasing sequence, then the series $\sum_{k=0}^{\infty} a_n \lambda_n$ is summable $|\bar{N}, p_n, \phi_n|_k, k \geq 1$.

3. The object of this paper is to generalize above theorem for $|\bar{N}, p_n, \phi_n, \delta, \beta|_k$ summability. However, we shall prove the following theorem.

Thoerem : Let $\{X_n\}$ be an almost increasing sequence and the sequences $\{\lambda_n\}$ and $\{p_n\}$ are such that the conditions (2.1)- (2.3) of above theorem are satisfied and

$$\sum_{n=1}^{m} \phi_n^{\beta(\delta k+k-1)} \left(\frac{p_n}{P_n}\right)^k |t_n|^k = 0(X_m) \text{ as } m \to \infty \qquad \ldots(3.1)$$

where $\{\phi_n\}$ be a sequence of positive real constants such that $\left\{\frac{\phi_n p_n}{P_n}\right\}$ is non-increasing sequence, then the series $\sum_{k=0}^{\infty} a_n$ is summable $|\bar{N}, p_n, \phi_n, \delta, \beta|_k \geq 1$, $\delta \geq 0$ and $\beta \geq 1$.

4. We need the following lemma for the proof of our theorem.

Lemma [7] : Under the condition on $\{X_n\}$ and $\{\lambda_n\}$ which are taken in the statement of our theorem, the following conditions hold:

(i) $nX_n |\Delta\lambda_n| = 0(1)$ as $n \to \infty$ …(4.1)

(ii) $\sum_{n=1}^{m} X_n |\Delta\lambda_n| < \infty$ …(4.2)

(iii) $X_n |\lambda_n| = 0(1)$ as $n \to \infty$ …(4.3)

PROOF OF THE THEOREM:

Let $\{T_n\}$ be the sequence of $(N,\ p_n)$ means of the series $\sum_{k=0}^{\infty} a_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n}\sum_{v=1}^{n} p_v s_v = \frac{1}{P_n}\sum_{v=1}^{n} p_v \sum_{z=0}^{n} a_v \lambda_z$$

$$= \frac{1}{P_n}\sum_{v=0}^{n} (p_n - p_{v-1})\, a_v \lambda_v$$

Then, for $n \geq 1$, we get

$$T_n - T_{n-1} = \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \lambda_v = \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1}\lambda_v v a_v}{v} \qquad \ldots(5.1)$$

Now applying Abel's transformation to the right hand side of (5.1), we get

$$T_n - T_{n-1} = \frac{(n+1)\,p_n t_n \lambda_n}{nP_n} - \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v a_v \lambda_v \frac{v+1}{v}$$

$$+ \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v a_v \Delta\lambda_v \frac{v+1}{v} + \frac{P_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v t_v \lambda_{v+1} \frac{1}{v}$$

$$= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} \qquad \text{say}$$

To complete the proof of the theorem, it is enough to show that

$$\sum_{n=1}^{\infty} \phi_n^{\beta(\delta k+k-1)} \, | T_{n,z} |^k < \infty \quad f\text{or } z = 1, 2, 3, 4 \qquad \ldots(5.2)$$

For we have

$$\sum_{n=1}^{m} \phi_n^{\beta(\delta k+k-1)} \, | T_{n,1} |^k = \sum_{n=1}^{\infty} \phi_n^{\beta(\delta k+k-1)} \left| \frac{(n+1)\,p_n t_n \lambda_n}{nP_n} \right|^k$$

$$= 0(1) \sum_{n=1}^{m} \phi_n^{\beta(\delta k+k-1)} \left(\frac{p_n}{P_n} \right)^k | t_n |^k \, | \lambda_n || \lambda_n |^{k-1}$$

$$= 0(1) \sum_{n=1}^{m} \phi_n^{\beta(\delta k+k-1)} \, | \lambda_n | \left(\frac{p_n}{P_n} \right)^k | t_n |^k$$

$$= 0(1) \sum_{n=1}^{m-1} \Delta | \lambda_n | \sum_{n=1}^{n} \phi_v^{\beta(\delta k+k-1)} \, | t_v |^k \left(\frac{p_n}{P_n} \right)^k$$

$$+ 0(1) | \lambda_m | \sum_{v=1}^{m} \phi_n^{\beta(\delta k+k-1)} \, | t_n |^k \left(\frac{p_v}{P_v} \right)^k$$

$$= 0(1) \sum_{n=1}^{m-1} \Delta | \lambda_n | X_n + 0(1) | \lambda_m || X_m | \qquad \text{by (3.1)}$$

$$= 0(1) \sum_{n=1}^{m-1} | \Delta\lambda_n | X_n + 0(1) | \lambda_m || X_m |$$

$$= 0(1) \quad \text{as} \quad m \to \infty$$

by condition (4.2) and (4.3) of lemma.

Again for k > 1 and applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, as in $|T_{n,1}|$, we have

$$\sum_{n=2}^{m+1} \phi_n^{\beta(\delta k+k-1)} |T_{n,2}|^k = \sum_{n=2}^{m+1} \phi_n^{\beta(\delta k+k-1)} \left| -\frac{p_n}{P_n P_{n-1}} \sum_{v=2}^{n-1} p_v t_v \lambda_v \frac{v+1}{v} \right|^k$$

$$= 0(1) \sum_{n=2}^{m+1} \phi_n^{\beta(\delta k+k-1)} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} p_v |t_v|^k |\lambda_v|^k \right\} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1}$$

$$= 0(1) \sum_{n=2}^{m+1} \phi_n^{\beta(\delta k+k-1)} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} p_v |t_v|^k |\lambda_v|^k \right\}$$

$$= 0(1) \sum_{v=1}^{m} p_v |t_v|^k |\lambda_v|^k \frac{1}{P_{n-1}} \left\{ \sum_{n=1}^{m} \phi_n^{\beta(\delta k+k-1)} \left(\frac{p_n}{P_n}\right)^k \right\} \frac{p_n}{P_n P_{n-1}}$$

$$= 0(1) \sum_{v=1}^{m} \left\{ (\phi_n)^{\beta(\delta k+k-1)} \left(\frac{p_v}{P_v}\right)^k \right\} p_v |t_v|^k |\lambda_v|^k \sum_{n=1}^{v+1} \frac{p_n}{P_n P_{n-1}}$$

$$= 0(1) \sum_{v=1}^{m} \phi_n{}^{\beta(\beta k+k-1)} |\lambda_v| \left(\frac{p_v}{P_v}\right)^k |t_v|^k$$

$$= 0(1) \sum_{v=1}^{m} \Delta|\lambda_v| \sum_{v=1}^{m} \phi_n{}^{\beta(\beta k+k-1)} |t_i|^k \left(\frac{p_i}{P_i}\right)^k$$

$$+ 0(1) |\lambda_v| \sum_{v=1}^{m} \phi_n{}^{\beta(\delta k+k-1)} |t_i|^k \left(\frac{p_i}{P_i}\right)^k$$

$$= 0(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + 0(1) |\lambda_m| X_m \qquad \text{by (3.1)}$$

$$= 0(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + 0(1) |\lambda_m| X_m$$

$$= 0(1) \quad \text{as } m \to \infty$$

by condition (4.2) and (4.3) of lemma.

Again we have

$$\sum_{n=2}^{m+1} \phi_n{}^{\beta(\delta k+k-1)} |T_{n,3}|^k$$

$$= 0(1)\sum_{n=2}^{m+1} \phi_n^{\beta(\delta k+k-1)} \left(\frac{p_n}{P_n}\right)^k \frac{1}{P_{n-1}} \times$$

$$\left\{\sum_{v=1}^{n-1} v|\Delta\lambda_v|^k \; p_v |t_v|^k\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right\}^{k-1}$$

$$= 0(1)\sum_{v=1}^{m} v|\Delta\lambda_v|^k \; p_v |t_v|^k \sum_{v=1}^{n-1} \left\{(\phi_n)^{\beta(\delta k+k-1)} \left(\frac{p_n}{P_n}\right)^k\right\} \frac{p_n}{P_n P_{n-1}}$$

$$= 0(1)\sum_{v=1}^{m} \left\{(\phi_v)^{\beta(\delta k+k-1)} \left(\frac{p_v}{P_v}\right)^k\right\} v|\Delta\lambda_v|^k \; p_v |t_v|^k \frac{1}{P_v}$$

$$= 0(1)\sum_{v=1}^{m-1} |\Delta(v|\Delta\lambda_v|)| \sum_{i=1}^{v} \phi_i^{\beta(\delta k+k-1)} |t_i|^k \left(\frac{p_i}{P_i}\right)^k$$

$$+ 0(1) m |\Delta\lambda_m| \sum_{i=1}^{m} \phi_i^{\beta(\delta k+k-1)} |t_i|^k \left(\frac{p_i}{P_i}\right)^k$$

$$= 0(1)\sum_{v=1}^{n-1} vX_v |\Delta^2\lambda_v| + 0(1)\sum_{v=1}^{m-1} X_v |\Delta\lambda_{v+1}| + 0(1) m |\Delta\lambda_m| X_m$$

$$= 0(1) \quad \text{as } m \to \infty \text{ by (2.3), (4.1) and (4.2)}$$

Finally, using the fact $P_n = O(np_n)$ by (2.1) as in $|T_{n,1)}|$, we have that

$$\sum_{n=1}^{m+1} \phi_n^{\beta(\delta k+k-1)} |T_{n,4}|^k = \sum_{n=1}^{m+1} \phi_n^{\beta(\delta k+k-1)} \left|\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v t_v \lambda_{v+1} \frac{1}{v}\right|^k$$

$$= 0(1)\sum_{v=1}^{m} \phi_n^{\beta(\delta k+k-1)} |\lambda_{n+1}| \left(\frac{p_v}{P_v}\right)^k |t_v|^k$$

$$= 0(1) \quad \text{as } m \to \infty$$

Therefore, we get

$$\sum_{n=1}^{\infty} \phi_n^{\beta(\delta k+k-1)} |T_{n,z}|^k < \infty \quad \text{for } z = 1, 2, 3, 4$$

This is the complete proof of our theorem.

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