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APPLICATIONS OF HAHN-BANACH THEOREM AND ABELIAN GROUP SUBADDITIVE WITH NORMED SPACE

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Leave Y alone a subgroup of an abelian bunch X and let $\mathbb{F}$ be a given assortment of subset of a straight space E over the rationals. Also, assume that $\mathbb{F}$ is a subadditive set-esteemed capacity characterized on X with values in $\mathbb{F}$. We build up certain conditions under which each added substance determination of the limitation of $\mathbb{F}$ to Y can be reached out to an added substance choice of $\mathbb{F}$. We additionally present a few utilizations of after effects of this sort to the soundness of useful conditions.

Keywords: Hahn-Banach theorem, normed space, Abelian group, Kuratowski-Zorn, subadditive.

INTRODUCTION

The Hahn-Banach hypothesis is an expansion hypothesis for straight functionals [I]. We will find in the following segment that the hypothesis ensures that a normed space is lavishly provided with limited straight functionals and makes conceivable a sufficient hypothesis of double spaces, which is a fundamental piece of the general hypothesis of normed spaces [II-VII]. Along these lines the Hahn-Banach hypothesis gets one of the most significant hypotheses regarding limited direct administrators. Besides, our conversation will show that the hypothesis likewise describes the degree to which estimations of a straight practical can be preassigned [VIII-XI]. The hypothesis was found by H. Hahn (1927), rediscovered in its current progressively broad structure by S. Banach (1929) and summed up to complex vector spaces by H. F. Bohnenblust and A. Sobczyk (1938). For the most part talking, in an expansion issue one considers a scientific item characterized on a subset $\mathbb{Z}$ of given set X and one needs to stretch out the article from $\mathbb{Z}$ to the whole set X so that specific essential properties of the item keep on holding for the all-inclusive item [XII-XV]. In the Hahn-Banach hypothesis, the item to be expanded is a direct useful f which is characterized on a subspace $\mathbb{Z}$ of a vector space X and has a specific boundedness property which will be defined as far as a sublinear utilitarian [XVI].

In the primary application, we show the presence of quadrature decides that are accurate for polynomials of degree at most n, depend on a lot of n +1 point, and have positive coefficients. Farkas' lemma is the key outcome supporting the direct programming duality and has assumed a focal job in the advancement of numerical streamlining. It is utilized in addition to other things in the confirmation of the Karush-Kuhn-Tucker hypothesis in nonlinear programming [XVII-XXII]. We give an application to an issue of best estimation from a curved cone in a Hilbert space. We will require a unique instance of the accompanying notable portrayal of best approximations from raised sets. As a simple outcome of a hypothesis describing is best approximations from a polyhedron [XXIII-XXV]. We give a class of issues identified with shape safeguarding guess that can be taken care of by Theorem.

II. Preliminaries

Throughout this paper, $\mathbb{R}, \mathbb{Q}$ and $\mathbb{Z}$ stand for the sets of all real, rational and integers, respectively. Our main goal is to give a generalization of the following well-known Hahn-Banach theorem:

THEOREM A. Let Y be a linear subspace of a real linear space X. Assume that $p : X \to \mathbb{R}$ is a functional such that

(i) $p(x+y) \le p(x) + p(y)$ for all $x, y \in X$;

(ii) $p(\alpha x) = \alpha p(x)$ for all $\alpha \ge 0$, and $x \in X$.

If $f : Y \to \mathbb{R}$ is a linear functional satisfying

$$f(x) \le p(x) \quad \text{for } x \in Y \qquad \dots(1)$$

then f can be extended to a linear functional $g : X \to \mathbb{R}$ with

$$g(x) \le p(x) \text{ for } x \in X \qquad \dots(2)$$

First we will rethink this hypothesis as far as a set-esteemed capacity with values in the group $cc(\mathbb{R})$ of all non-vacant, smaller, arched subsets of $\mathbb{R}$. Plainly, the components of $cc(\mathbb{R})$ are only the non-void minimal interims in $\mathbb{R}$. Watchwords and expressions: Hahn-Banach hypothesis, added substance choice, subadditive set-esteemed capacity, parallel crossing point property, security of utilitarian conditions. A s.v. work F mapping X into $cc(\mathbb{R})$ is said to be subadditive iff

$$F(x+y) \subset F(x) + F(y) \text{ for all } x, y \in X \qquad \dots(3)$$

and it is called positively homogeneous iff

$$F(\alpha x) = \alpha F(x) \text{ for all } \alpha \ge 0 \text{ and } x \in X \qquad \dots(4)$$

Addition of sets and multiplication of sets by scalars are here understood in the Minkowski sense, *i.e.*

$A+B:=\{a+b:a\in A,b\in B\}$, $\alpha A:=\{\alpha a;a\in A\}$ for any $A,B\subset\mathbb{R}$ and $\alpha\in\mathbb{R}$.

One can easily check that inequalities (1) and (2) of Theorem A are equivalent to

$$-p(-x)\le f(x)\le p(x) \text{ for } \quad x\in Y \qquad \ldots(5)$$

$$-p(-x)\le g(x)\le p(x) \text{ for } x\in X \qquad \ldots(6)$$

respectively. Moreover, by (i) and (ii) we have p (0)=0 and

$$-p(-x)\le p(x) \text{ for } x\in X$$

Therefore, we may correctly define an s.v. function $F:X\to cc(\mathbb{R})$ by

$$F(x):=[-p(-x),p(x)] \text{ for } x\in X \qquad \ldots(7)$$

It is evident that *F* is subadditive, positively homogeneous and odd, *i.e.*

$F(-x)=-F(x)$ for all $x\in X$. Conversely, each s.v. function $F:X\to cc(\mathbb{R})$ which is subadditive, positively homogeneous and odd must be of the form (7) with a functional $p:X\to R$ satisfying (i) and (ii). Now Theorem A may be interpreted as a result on extending partial additive selections of an s.v. function $F:X\to cc(\mathbb{R})$, as follows:

THEOREM B. Let Y be a linear subspace of a real linear space *X*. Assume that

$F:X\to cc(\mathbb{R})$ is a subadditive, positively homogeneous and odd s.v. function. If $f:Y\to\mathbb{R}$ a linear functional such that

$$f(x)\in F(x) \text{ for all } \quad x\in Y$$

then *f* extends to a linear functional *g* defined on the whole of *X* and such that

$$g(x)\in F(x) \text{ for all } x\in X$$

In the following segment we sum up Theorem B to the accompanying conceptual setting. Rather than the straight space *X* we consider a discretionary abelian bunch $(X,+)$, and the family $cc(\mathbb{R})$ is supplanted by an aphoristically given assortment $\mathbb{F}$ of subsets of a direct space *E* over $\mathbb{Q}$. Among the suspicions forced on $\mathbb{F}$ the essential job is played by the alleged paired crossing point property. It implies that each subfamily of $\mathbb{F}$, any two individuals from which cross, has a non-void convergence. This property was first presented and contemplated. It is notable that the assortment of all non-void conservative interims in *R* has the two fold crossing point property.

II. Generalizations of the Hahn-Banach Theorem

Since we now assume that X is a group (not a linear space), it is natural to discuss additive (instead of linear) selections of an s.v. function $F : X \to \mathbb{F}$, *i.e*, functions $g : X \to E$ such that

$$g(x+y) = g(x) + g(y) \text{ for all } x, y \in X$$

$$g(x) \in F(x) \text{ for all } x \in X$$

THEOREM 1. Let Y be a subgroup of an abelian group $(X,+)$ and let E be a linear space over $\mathbb{Q}$. Furthermore, let $\mathbb{F}$ be a family of non-empty subsets of E having the binary intersection property and satisfying the following conditions:

$$A \in \mathbb{F},\ v \in E \Rightarrow A + v \in \mathbb{F} \qquad \ldots(8)$$

$$A \in \mathbb{F},\ n \in Z^* := Z \setminus \{0\} \Rightarrow \frac{1}{n} A \in \mathbb{F} \qquad \ldots(9)$$

Assume that $F:X \to \mathbb{F}$ is a subadditive s.v. function such that

$$F(nx) \subset nF(x) \text{ for all } x \in X \text{ and } n \in Z^* \qquad \ldots(10)$$

If $f : Y \to E$ is an additive selection of the restriction of F to Y (denoted by $F|_Y$), then f can be extended to an additive selection of F.

Proof. Denote by Ω the family of all additive maps such that

$$Y \subset dom\varphi \subset X, dom\varphi \text{ is a subgroup of } X, \varphi(x) \in F(x) \text{ for } x \in dom\varphi$$

and $$\varphi(x) = f(x) \text{ for } x \in Y.$$

The family Ω is partially ordered by the relation $\prec$ defined by

$$\varphi \prec \psi \text{ iff } dom\varphi \subset dom\psi \text{ and } \phi = \psi / dom\phi$$

It is easy to see that every chain $C \subset \Omega$ has an upper bound in Ω : it is the map ϕ_c such that $dom\phi_c := \{dom\phi : \phi \in C\}$ and $\phi / dom\phi = \phi$ for each $\phi \in C$.

By the Kuratowski-Zorn lemma,

Ω contains at least one maximal element g.

To complete the proof it is enough to show that $dom\, g = X$.

Suppose that there exists a $z_0 \in X \setminus dom\, g$ and put

$$W := \{x + nz_0 : x \in dom\, g, n \in Z\}$$

Obviously W is a subgroup of X properly containing $dom\, g$. We distinguish two cases depending on whether the set $A := \{k \in Z : kz \in dom\, g\}$ is empty or not.

Case 1: $A \neq \theta$. If $k, l \in A$, then $k, l \in A$

and $$l\, g(kz) = g(lkz) = kg(lz),$$

hence $$g(kz)/k = g(lz)/l$$

Putting

$$u_0 := g(kz)/k \text{ for some } k \in A.$$

We define an element $u_0 \in E$ which does not depend on the choice of $k \in A$.

Next we define $g : W \to E$

$$\bar{g}\,(x + nz) := g(x) + nu_0 \quad \text{for } x \in dom\, g \text{ and } n \in Z. \qquad \text{...(11)}$$

If an element of W admits two representations:

$x + nz = y = mz$ with some $x, y \in dom\, g$ and $n, m \in Z$, then $(m-n)z = x - y \in dom\, g$. Then there are two possibilities: either $m = n$ or $m - n \in A$. In the first case we have $x = y$ and

$g(x) + nu_0 = g(y) + mu_0$. If the second possibility holds , then

$$u_0 = \frac{g(m-n)z}{m-n}$$

which implies that

$$g(x) - g(y) = g(x-y) = g(m-n)z = (m-n)u_0$$

and consequently, $g(x) + nu_0 = g(y) + mu_0$.

Thus the definition of $\bar{g}$ is correct. It is also clear that $\bar{g}$ is additive and $\bar{g}\,|_{dom\, g} = g$.

Now let $x \in dom\, g, n \in Z$ and $k \in A$.. Then

$$\bar{g}\,(x + nz_0) = g(x) + nu_0$$

$$= g(x) + \frac{ng(kz)}{k} = \frac{gk(x+nz)}{k} \in \frac{1}{k} F(k(x+nz)) \subset F(x+nz)$$

Thus $\bar{g}$ is an additive selection of $F|_w$, contrary to the maximality of g in Ω.

Case 2: $A=\theta$. Then $kz_0 \in X \setminus dom\, g$ for every $k \in \mathbb{Z}^*$ and $n,m \in Z^*$. By the subadditively of F and by (11) we have

$$mg(x)-ng(y)=g(mx-ny)\in F(mx-ny)$$

$$=F(mx+nmz_0-nmz_0-ny)$$

$$\subset F(m(x+nz_0))+F(-n(y+mz_0))$$

$$\subset mF(x+nz_0)-nF(y+mz_0)$$

Consequently ,

$$0\in m[F(x+nz_0)-g(x)]-n[F(y+mz_0)-g(y)]$$

which means that

$$0\in \frac{1}{n}[F(x+nz_0)-g(x)]-\frac{1}{m}[F(y+mz_0)-g(y)]$$

We conclude that for any $x,y\in dom\, g$ and $n,m\in \mathbb{Z}^*$ the intersection

$$\frac{1}{n}[F(x+nz_0)-g(x)]\bigcap \frac{1}{m}[F(y+mz_0)-g(y)]$$

Is non-void. From the hypotheses it now follows that

$$\bigcap\{\frac{1}{n}[F(x+nz_0)-g(x): x\in dom\, g, n\in Z\ \}\neq 0$$

Let u_0 be in this intersection; then $g(x)+nu_0 \in F(x+nz_0)$ for all $x\in domg$ and $n\in Z$. Similarly to **Case 1** we define $\bar{g}:W\to E$ by (11). This definition is unambiguous, since now

$x+nz_0=y+mz_0$ (with $x,y\in domg$ and $n,m\in Z$) only holds if $x=y$ and $n=m$. Moreover , $\bar{g}$ is an additive selection of $F|_W$, which again contradicts the maximality of g in Ω.

The proof is finished.

If members of $\mathbb{F}$ are $\mathbb{Q}$-convex, *i.e.* $\alpha A+(1-\alpha)A\subset A$ for all $\alpha\in Q\bigcap[0,1]$ and $A\in\mathbb{F}$, then assumption (10) on the s.v function $F:X\to\mathbb{F}$ can be weakened.

THEOREM 2. Let Y be a subgroup of an abelian group $(X,+)$ and let E be a linear space over $\mathbb{Q}$. Moreover , let $\mathbb{F}$ be a family of non-empty $\mathbb{Q}$-convex subsets of E having the binary intersection property and satisfying conditions (8) and (9) of Theorem 1. If $F:X\to\mathbb{F}$ a subadditives.v. function such that

$$F(-x) \subset -F(x) \text{ for all } x \in X \qquad \ldots(12)$$

then every additive selection of $F|_Y$ has an extension to an additive selection of *F*.

Proof. It is sufficient to observe that in fact *F* satisfies (10). Indeed, if $x \in X$, $n \in Z$ and $n > 0$, then by the subadditivity of *F* and by the $\mathbb{Q}$-convex of *F*(*x*), we derive

$$F(nx) \subset F(x) + ... + F(x) \subset nF(x)$$

If $n \in Z$ and *n*>0, then on account of (12) we have

$$F(nx) \subset -F(-nx) \subset -(-n)F(x) = nF(x)$$

which completes the proof.

The next result is an immediate consequence of Theorems 1 and 2 with

$$Y := \{0\} \text{ and } f(0) := 0$$

COROLLARY 1. Let $(X,+)$ be an abelian group and let *E* be a linear space over $\mathbb{Q}$. Under the hypotheses of either Theorem 1 or Theorem 2 concerning the family $\mathbb{F}$ and the s.v. function $F : X \to \mathbb{F}$, then *F* has an additive selection. The collection $cc(\mathbb{R})$ is a simple example of a family $\mathbb{F}$ satisfying all the conditions in both Theorems 1 and 2. If *X*, *Y* and $F : X \to cc(\mathbb{R})$ satisfy all the assumptions of Theorem *B*, then by virtue of either Theorem 1 or 2 a given linear selection $f : Y \to \mathbb{R}$ of $F|_Y$ can be extended to an additive (a prior not necessarily linear) selection $g : X \to \mathbb{R}$ of *F*. With each $x \in X$ we associate a function $g_x : \mathbb{R} \to \mathbb{R}$ defined by

$$g_x(\alpha) := g(ax) \text{ for } \alpha \in R$$

which is additive and

$$g_x(\alpha) \le \sup F(\alpha x) = \alpha \sup F(x) \text{ for } \alpha \ge 0.$$

In particular, g_x is upper bounded on a non-empty open interval and by a classical result. Theorem 1 and the subsequent remarks) it has the form

$$g_x(\alpha) = \alpha g_x(1) \text{ for all } \alpha \in \mathbb{R}$$

This assures that *g* is homogeneous and shows that Theorem *B* may be easily deduced from both Theorem 1 and 2.

II. Main Results

THEOREM 3. Let *Y* be a subgroup of an abelian group $(X,+)$ and let *K* be a linear subspace of a linear space *E* over $\mathbb{Q}$. If $\phi : X \to E$ is such that

$$\phi(x+y) - \phi(x) - \phi(y) \in K \text{ for all } x, y \in X \qquad \ldots(13)$$

and $f : Y \to E$ is an additive map satisfying

$$f(x)-\phi(x)\in K \text{ for all } x\in X$$

then *f* can be extended to an additive function $g: X \to E$ such that

$$g(x)-\phi(x)\in K \text{ for all } x\in X$$

Proof. First we observe that the family

$$F := \{w+K : w\in E\} \text{ has the binary intersection property}$$

(in fact , every subfamily of $\mathbb{F}$ any two of whose members intersect consists of a single set) . Clearly, all elements of $\mathbb{F}$ are $\mathbb{Q}$ -convex and $\mathbb{F}$ satisfies (6) and (7). We consider an s.v. function $F: X \to \mathbb{F}$ given by

$$F(x) := \phi(x)+K \text{ for all } x\in X$$

It is evidently subadditive and $f: Y \to E$ is an additive selection of $F|_Y$.

To check that *F* satisfies (12) set $x = y = 0$ in (10), hence $\phi(0)\in K$.

Moreover, setting $y=-x$ in (12) we get $\phi(0)-\phi(x)-\phi(-x)\in K$, which combined with the preceding relation implies that

$$F(-x)=\phi(-x)+K \subset -\phi(x)+K = -F(x) \text{ for } x\in X$$

Now the conclusion follows directly from Theorem 2.

The subsequent corollary was first established in a different way. It results from our Theorem 3 upon setting $Y := \{0\}$

COROLLARY 2. Let (*X*, +) be an abelian group and let *K* be a linear subspace of a linear space *E* over $\mathbb{Q}$. If $\phi: X \to E$ satisfies (12), then there exists an additive function $g: X \to E$ such that

$$g(x)-\phi(x)\in K \text{ for all } x\in X$$

In the sequel we shall say that a normed space $(E, \|\ \|)$ has the binary intersection property iff the collection of all closed balls in *E* has the binary intersection property in the same introduced before. We will be concerned with the following inequality:

$$\|\phi(x+y)-\phi(x)-\phi(y)\| \le r(x)+r(y)-r(x+y) \qquad \ldots(14)$$

for $x, y\in X$,where $(X,+)$ is an abelian group, ϕ maps X into *E* and *r* is

a real –valued subadditive function on *X*. A study of this inequality with the "control function'' $r := \|\ \|$ was first proposed by *D*. Yost (cf. [6] and [7]) and then it was undertaken by connection with some stability questions for functional equations. Here, we prove the following extension theorem:

THEOREM 4. Let Y be a subgroup of an abelian group $(X,+)$ and let $(E,\| \,\|)$ be a normed space having the binary intersection property. Moreover, suppose that $r : X \to [0,\infty)$ is an even, subadditive function and $\phi : X \to E$ is an odd map satisfying (11). If $f : Y \to E$ is an additive function such that

$$\| f(x) - \phi(x) \| \le r(x) \text{ for } x \in Y \qquad \ldots(15)$$

then *f* has an extension to an additive function $g : X \to E$ such that

$$\| g(x) - \phi(x) \| \le r(x) \text{ for all } x \in X \;.$$

Proof. For $v \in E$ and $\rho \in [0,\infty)$ let $K(v,\rho)$ denote the closed ball in E with center v and radius ρ. Then $\mathbb{F} := \{K(v,\rho), v \in E,\ \rho \in [0,\infty)\}$ is a family of convex sets which, by hypothesis, has the binary intersection and, evidently, satisfies (8) and (9).

Notice that for any $\rho_1, \rho_2 \in [0,\infty)$ we have

$$K(0,\rho_1) + K(0,\rho_2) = K(0,\rho_1 + \rho_2).$$

Indeed, the inclusion $\subset$ is clear .

Conversely , if $w \in K(0,\rho_1 + \rho_2)$, then $w = u + v$, where

$$u := \frac{\rho_1}{\rho_1 + \rho_2} w,\ u := \frac{\rho_2}{\rho_1 + \rho_2} w$$

(without loss of generality one may assume that $\rho_1 > 0$ and $\rho_2 > 0$).

Inequality (11) may be written as

$$\phi(x+y) - \phi(x) - \phi(y) \in K(0, r(x) + r(y) - r(x+y))$$

which yields

$$\phi(x+y) - \phi(x) - \phi(y) + K(0, r(x+y))$$

$$\subset K(0, r(x) + r(y) - r(x+y)) + K(0, r(x+y)$$

$$= K(0, r(x) + r(y)) = K(0, r(x)) + K(0, r(y))$$

Hence

$$\phi(x+y) + K(0, r(x+y)) \subset \phi(x) + K(0, r(x)) + \phi(y) + K(0, r(y))$$

or equivalently,

$$K(\phi(x+y), r(x+y)) \subset K(\phi(x), r(x)) + K(\phi(y), r(y))$$

Now we define an s.v. function $F : X \to \mathbb{F}$ by

$$F(x) := K(\phi(x), r(x)) \text{ for } x \in X$$

We have just shown that F is subadditive. It is also odd, because

$$F(-x) = K(\phi(-x), r(-x)) = K(-\phi(x), r(x))$$

$$= -K(\phi(x), r(x)) = -F(x) \text{ for } x \in X$$

Moreover, on account of (14), f is a selection of $F|_Y$. Applying Theorem 2 we can extend *f* to an additive selection *g* of *F*. In particular, *g* satisfies (15) and the proof is finished.

COROLLARY 3. Under the assumptions of Theorem 4 on X, E, r and ϕ there exists an additive function $g : X \to E$ such that condition (15) holds true.

Proof. We can use Theorem 4 with $Y := \{0\}$ and $f(0) := 0$ since (13) guarantees that

$$\| f(0) - \phi(0) \| = \| \phi(0) \| \le r(0)$$

COROLLARY 4. Suppose that the hypotheses of Theorem 4 on X, E, r and ϕ are satisfied except that ϕ does not have to be odd. Then there exists an additive function $g : X \to E$ such that

$$\| g(x) - \phi(x) \| \le 2r(x) \text{ for } x \in X$$

Let ϕ_e and ϕ_o stand for the even and odd part of ϕ, respectively,

i.e. $$\phi_e(x) = \frac{1}{2}(\phi(x) + \phi(-x)), \quad \phi_o(x) = \frac{1}{2}(\phi(x) - \phi(-x))$$

for $x \in X$. From (11) we infer that $\| \varphi(0) \| \le r(0)$ and

$$\| \phi_e(x) \| - \frac{1}{2}\| \phi(0) \| \le \| \phi_e(x) - \frac{1}{2}\phi(0) \| = \frac{1}{2}\| \phi(x) + \phi(-x) - \phi(0) \|$$

$$\le \frac{1}{2}(r(x) + r(-x) - r(0))$$

$$= r(x) - \frac{1}{2}r(0) \text{ for } x \in X$$

Hence

$$\| \phi_e(x) \| \le r(x) + \frac{1}{2}(\| \phi(0) \| - r(0)) \le r(x) \quad \textit{for} \quad x \in X$$

Moreover, the odd part of ϕ also satisfies (11):

$$\|\phi_o(x+y)-\phi_o(x)-\phi_o(y)\|$$

$$=\frac{1}{2}\|\phi(x+y)-\phi(-x-y)-\phi(x)+\phi(-x)-\phi(y)+\phi(-y)\|$$

$$\leq\frac{1}{2}(\|\phi(x+y)-\phi(x)-\phi(y)\|+\|\phi(-x)+\phi(-y)-\phi(-x-y)\|$$

$$\leq\frac{1}{2}(r(x)+r(y)-r(x+y)+r(-x)+r(-y)-r(-x-y)\|)$$

$$=r(x)+r(y)-r(x+y) \text{ for } x,y\in X$$

Therefore, by Corollary 3, one can find an additive function $g: X\to E$ such that

$$\|g(x)-\phi_o\|\leq r(x) \text{ for } x\in X$$

Finally , we have

$$\|g(x)-\phi(x)\|=\|g(x)-\phi_o(x)-\phi_e(x)\|$$

$$\leq\|g(x)-\phi_o(x)\|+\|\phi_e(x)\|\leq 2r(x) \quad \text{for } x\in X$$

which was to be shown.

A result similar to our Corollary 4 was proved, where X was assumed to be an amenable group and the technique of invariant means was used.

COROLLARY 5. Let $(E,\|\ \|)$ be a normed space which may be equipped with a new norm $\|\ \|_0$ equivalent to $\|\ \|$ and such that $(E,\|\ \|_0)$ has the binary intersection property. Suppose that X, r and ϕ satisfy the same assumptions as in Corollary 4. Then there exists an additive function $g: X\to E$ such that

$$\|g(x)-\phi(x)\|\leq 2\alpha\beta r(x) \text{ for } x\in X$$

where α and β are positive constants with $\|u\|_0\leq\alpha\|u\|$ and $\|u\|\leq\beta\|u\|_0$ for all $u\in E$.

Proof. If we put $r_0(x):=\alpha r(x)$, then

$$\|\phi(x+y)-\phi(x)-\phi(y)\|_0\leq\alpha\|\phi(x+y)-\phi(x)-\phi(y)\|$$

$$\leq\alpha(r(x)+r(y)-r(x+y))=r_0(x)+r_0(y)-r_0(x+y)$$

for $x,y\in X$. By virtue of Corollary 4 there exists an additive function $g: X\to E$ such that

$$\|g(x)-\phi(x)\|_0 \le 2r_0(x) = 2\alpha r(x) \text{ for } x \in X$$

Hence

$$\|g(x)-\phi(x)\| \le \beta\|g(x)-\phi(x)\|_0 \le 2\alpha\beta r(x) \text{ for } x \in X$$

which completes the proof.

CONCLUSION

We may prove the following result on extending additive maps which approximate a function with Cauchy differences in a given linear space K over $\mathbb{Q}$. We show that a single special separation theorem namely, a consequence of the geometric form of the Hahn-Banach theorem can be used to prove Farkas type theorems, existence theorems for numerical quadrature with positive coefficients, and detailed characterizations of best approximations from certain important cones in Hilbert space.

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