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SYMMETRIES AND CONSERVATION LAWS

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Various external and internal symmetries (exact and approximate) of physical system have been discussed and various observed conservation laws have been elaborated. The connection between these two apparently uncorrelated properties of physical systems (*i.e.* the symmetries and conservation laws) have been analyzed classically as well quantum mechanically.

INTRODUCTION

Perplexed and troubled by the apparent diversities and complexities of Nature, man at an early stage of his awakening conceived the notion of ultimate harmony and symmetry of this universe. It is the very idea of symmetries which enables us, right upto the present day, to bring order into the most sophisticated complex phenomena. Symmetry is one idea by which man, through ages, has tried to comprehend and create order, beauty and perfection. The simplicity which the physicists have come to expect of Nature, has been sought almost exclusively in terms of symmetries rather than detailed dynamics. In many cases the use of symmetries is essential either because the system is complicated and we cannot perform the exact calculations easily or because a consistent dynamical theory of the concerned phenomena does not exist. The generalization of symmetriesand Gellman's totalitarian principle: 'Anything not formidable is compulsory to exist'; gave rise tomany discoveries in physics in generaland particle physics in particular, starting from neutrinos to galaxies.

A remarkable fact about all the physical systems is the symmetries or the invariances they possess under certain transformations. Some of these symmetries are exact and some are only approximate. When we refer to a symmetry as exact, we mean that no violation has been so far observed. It may be possible that future experiment will show that a symmetry now thought as

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exact is really only approximate. The examples of exact external symmetry (based on space and time) are invariance of a physical system under rotations and translations in space and time due to the isotropy and homogeneity of space and time. An example of an approximate symmetry is the symmetry under space reflection. Another striking fact about nature is that among many properties of physical systems that continuously change with time, a few properties remain constant. These constant properties appear into many different physical systems and they are among most fundamental laws of physics and are known as the conservation laws. In addition to those properties which, so far as we know from experiments, are exactly conserved, there are other properties which are only approximately conserved. The oldest known exact conservation laws are those of linear momentum, angular momentum and energy. A familiar law which holds only approximately is the conservation of Parity. In the present paper we will examine the connection between these two apparently uncorrelated properties of physical systems (*i.e.* the symmetries and conservation laws) classically as well quantum mechanically.

1. The Link Between External Symmetries and Conservation Laws

The connection between these two apparently disconnected and uncorrelated properties of physical systems was first noticed in classical mechanics by Jacobi. He showed that for a classical system describable by classical Lagrangian, the invariance of Lagrangian under spatial translation implies that the linear momentum is conserved and its invariance under rotation implies the conservation of angular momentum. A little later Schuz [1] derived the principle of conservation of energy from the invariance of Lagrangian under the time translation. Thus we have

(i) Invariance of $L = T - V$ $Lagrangian$ Under Space Translation

$$\Updownarrow$$

Conservation of Linear Momentum

(ii) Invariance of $Lagrangian$ Under Rotation

$$\Updownarrow$$

Conservation of Angular Momentum

(iii) Invariance of L Under Time Translation

$$\Updownarrow$$

Conservation of Energy

For a conservative system we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0,$$

and $$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

Thus $\dot{p_J} = \frac{dp_j}{dt} = \frac{\partial L}{\partial q_j}$

Hence the invariance of Lagrangian under the translation in generalized coordinate q_j *i.e.*

$$\frac{\partial L}{\partial q_j} = 0$$

implies

$$\dot{p_J} = \frac{dp_j}{dt} = 0$$

⇒ **Conservation of Linear Momentum**

Angular Momentum is Generalized momentum Conjugate to Generalized Coordinate

$$q_j = \delta\vec{\varphi} = \hat{n}\delta\varphi$$

$$\Rightarrow \frac{\partial L}{\partial \dot{q_J}} = \sum_i \hat{n}\,.(\vec{r_i} \times m_i\vec{v_\imath}) = \hat{n}\,.\vec{L}$$

$$\frac{\partial L}{\partial q_j} = 0 \ \Rightarrow \frac{d(\hat{n}\,.\vec{L}\,)}{dt} = 0 \qquad \text{(\textbf{Rotational Invariance})}$$

⇒ $\frac{d\vec{L}}{dt} = 0$ ⇒ **Conservation of Angular momentum**

Energy is generalized momentum conjugate to time

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dE}{dt} = 0$$

Invariance Under time Translation ⇒ **Energy Conservation**

In classical and quantum mechanics, the conservation of linear momentum, and angular momentum and energy follows from the properties of Hamiltonian under spatial translations, rotations and the time-translation. In Hamiltonian formulation the equal status is given to Coordinates and Momentum

$$\Downarrow$$

6N-Dimensional Space (Phase space) (6N Partial Differential Equation of first order)

Classically we have the Hamiltonian as

$$H = \sum_i p_i\dot{q_\imath} - L; \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

$$\frac{\partial H}{\partial q_j} = 0 \Rightarrow \frac{dp_j}{dt} = 0 \text{ Conservation Laws}$$

2. Quantum Mechanically:

$$\frac{d}{dt} < \hat{A} >_\beta = < \frac{[\,\hat{A}, \widehat{H}]}{i\hbar} >_\beta \ + < \frac{\partial \hat{A}}{\partial t} >_\beta$$

$$\Downarrow$$

For $$\frac{\partial \hat{A}}{\partial t} = 0;\ \left[\hat{A}, \hat{H}\right] = 0 \Rightarrow \frac{d}{dt} < \hat{A} >_{\beta} = 0$$

$$\Downarrow$$

Conservation of observable A

Herglotz gave the complete discussion [2] of the ten constants of motion associated with the invariance of the Lagrangian under the group of inhomogeneous Lorentz transformations (three rotations about three cartesian coordinates, three Lorentz transformations corresponding to them, and the four translations in space and time).

Thus Invariance under Inhomogeneous Lorentz Transformations

$$\Downarrow$$

$$\text{Ten Constants of Motion } (\, p_j;\ L_j\ ;\ K_j\ ; E) \qquad \text{.... (1)}$$

where K_j are the generalized momentum conjugate to cyclic coordinates associated with three Lorentz transformations.

3. Complex Angular Momentum Operator

Ignoring the translations (spatial as well as temporal), the remaining generators of the set given by eqn. (1) give the six homogeneous transformations which constitute Homogeneous Lorentz Group (HLG). In the light of several new developments in particle physics in last decade, there has been a new interest in the study of HLG and the universal covering group SL(2,C), which play important roles in the study of various problems like dual amplitudes in the Koba-Neilsen form [3,4,5]; harmonic analysis of scattering amplitudes [6]; Regge classification of hadrons [7] and many supersymmetric formulations [8]. Denoting by L_j and K_j(j = 1, 2, 3) the generators of purerotations and space-time rotations (pure Lorentz transformations), we have already shown [9] that the linear combinations

$$Z_{ij} = \frac{1}{2}[L_i + iK_j] \qquad \text{... (2)}$$

and $$X_{ij} = \frac{1}{2}[L_i - iK_j] \qquad \text{... (3)}$$

Give the generalized generators of the complex angular momentum operators associated with the collective relative motion of a body rotating about i^{th}-axis and moving with relativistic velocity along j^{th}-axis. For i=j these generators may be written as

$$Z_{jj} = Z_j = \frac{1}{2}[L_j + iK_j] \qquad \text{.... (4)}$$

and $$X_{jj} = X_j = \frac{1}{2}[L_j - iK_j] \qquad \text{... (5)}$$

which satisfy the Lie algebra of two independent angular momentum operators in complex spaces (L, K) and (L, – K) and they have therefore been defined as the components of complex angular momentum in these spaces. Since these operators Z_j and X_j commute, the representations of HLG can be considered as the direct product of two groups generated by these operators, as has already been shown by Smorodinski and Huszar [10].

Combining the components of Z_1 and Z_2 in the following manner

$$Z_+ = Z_1 + iZ_2 \quad .... (6)$$

and

$$Z_- = Z_1 - iZ_2, \quad ... (7)$$

It can be shown that the space R in which the generators of HLG may be analysed into a linear sum of invariant spaces R_l, in each of which an irreducible representation of weight l of the group of ordinary rotations is obtained. These equation (4), (5), (6) and (7) give the compact operator formulation to reformulate the Gel'fand-Naimark [8] theory of representation of the SL (2, C) group which is the universal covering group of HLG. The representation of the components of complex angular momentum operators constructed by eqn. (4) has already been under taken in our earlier paper [11] in the canonical basis of their Eigen vectors. In our another earlier paper [12], we have used the generators of complex angular momentum in complex space and derived the realizations of HLG for nonzero mass, zero mass and imaginary mass systems.

4. Nothern's Theorem

The general connection between symmetries and the conservation laws is given in terms of Nother's theorem, which essentially states that every conservation law is the consequence of a symmetry of the physical system. In other words, whenever a conservation law holds for a physical system, the Hamiltonian of the system is invariant under the corresponding group of transformations. Its converse is not always true:

Time reversal $t \to -t$ does not lead to conservation law.

In general also even if the system has a Hamiltonian which is invariant under a group of transformations, there may not be a conservation law. Then the question arises that what types of symmetries do and what types do not imply the conservation laws. This question was answered by Wigner [13] by showing that all symmetry transformations of a quantum mechanical state can be chosen so as to correspond to either unitary or anti-unitary operators

$$\mathrm{I}\alpha, t > = \cup\, (t.t_0)\, \mathrm{I}\alpha, 0 > \Rightarrow \cup^{\dagger}\cup = \cup\, \cup^{\dagger} = \pm I \quad ... (8)$$

He also demonstrated that it is the unitary transformation which is associated with a conservation law. Since time reversal operator is anti-unitary, the invariance of the physical systems under time reversal does not lead to a conservation law.

A set of symmetry transformations of a physical system has the mathematical propertiesthat are associated with a group. These constitute symmetry groups like rotation group, translation group, reflection group etc. Rotations and translations in space and time may be made through any angle or through any distance (spatial or temporal). For this reason there are a continuous infinity of transformations which leave certain physical systems invariant. These transformations correspond to continuous groups. However, there is one basic difference between the rotations and translations. The rotations vary over a finite angular domain while the translations in space and time vary over an infinite domain and hence the corresponding groups have many different properties.

All the states of a physical system, which can be obtained by operating with all unitary operators $\widehat{U}$ on a given state, can be written as linear combinations of a set of basic vectors which span the subspace of eigen states of Hamiltonian with the given energy. These vectors are the basis vectors of the unitary representation operator $\widehat{U}$of the group of transformations. In general, these vectors are the basis vectors of an irreducible representation. The basis vectors of an irreducible representation of a symmetry transformation denote a set of quantum mechanical states. The symmetries may be finite or infinitesimal ones. Infinitesimal symmetries are those in which associated alternations are infinitesimal. Constants of motion resulting from this sort of symmetries have classical counter parts. It may be shown classically that if $\widehat{G}$ is generator of an infinitesimal canonical transformation that leave the Hamiltonian of the system invariant, then $\widehat{G}$ is the constant of motion (leading to a conservation law). In quantum mechanics, the analogous situation is of particular interest when the infinitesimal character is expressed through linear dependence on an infinitesimal numerical parameter ∈:

If $\hat{U} = \hat{I} + i \in \hat{F}$ and $\hat{F}$ is Hermitian,

then $$[\hat{U}, \hat{H}] = 0 \Rightarrow F \text{ is Constant of Motion} \qquad \text{... (9)}$$

In addition to infinitesimal symmetries there are also finite symmetries. Unlike classical mechanics, in quantum mechanics such symmetries are significant because if the operator defining such symmetry is Hermitian, then it corresponds to an observable quantity. Even if it is not Hermitian and therefore non-observable, its existence as constant of motion may facilitate the search for Eigen states of the system.

5. Internal Symmetries

Besides the space-time symmetries (*i.e.* external symmetries), there are the symmetries connected with invariance of Hamiltonian under those transformations which do not involve space and time coordinates. These symmetries are known as internal ones, like rotations in spin and Isospin spaces. These symmetries are much more **Mysterious** than the external symmetries**.** There are several following questions associated with these symmetries:

Why Should Baryon Number be Conserved?

Why Are Proton and Neutron so Similar?

Why Should Approximate Symmeries Apply to Hadrons?

What Relates Internal Quantum Numbers to Space-Time Transformations?

Why Some Symmetries are Exact and Some Approximate?

Reply to these questions and the additional understanding of these symmetries comes from the study of local Gauge Transformations which relates apparently internal quantum numbers to space time dependent transformations.

Gauge Transformations :

Hermann Weyl gave these transformation in 1919 (Good Idea Born Before Time). It could survive as Symmetry of Maxwell's Equations (First Unification):

$$\vec{H} = \nabla \times \overrightarrow{A} \; ; \; \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

$$\Updownarrow$$

$$V \to V' = V - \frac{\partial \chi}{\partial t} \; ; \vec{A} \to \overrightarrow{A'} = \vec{A} + \nabla \chi$$

$\Rightarrow$ $$A^{\mu} \to A^{\mu} - \partial^{\mu} \chi \qquad \text{... (10)}$$

Global And Local Gauge Theories:

Under local Gauge Transformation :$\Psi(x,t) \to \Psi'(x,t) = exp\{ie\chi(x,t)\}\Psi(x,t)$

Invariance of Q. E Requires $(D^{\mu}\Psi) \to exp\{ie\chi(x,t)\}(D^{\mu}\Psi)$

where $D^{\mu} = \partial^{\mu} + ieA^{\mu}$ (Covariant Derivative) ... (11)

$\Rightarrow$ Space-time Phase Invariance Demands Interacting Gauge Field

Interaction is Mediated by Massless Gauge Boson

$$\Downarrow$$

Photon in QED :$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$: U(1) Gauge Group

$W^{\pm}; Z^0$in Y.M (QFT): $F^{\mu\nu} = W^{\nu,\mu} - W^{\mu,\nu} + g(W^{\mu} \times W^{\nu})$: SU(2) Gauge Group

Eight Gluons In QCD: SU$(3)_c$ = SU(2) × U(1) Gauge Group ... (12)

$$\Downarrow$$

Nature of Force :	Range:	Strength:	Gauge Group:	Gauge Bosons (Mediators):	
E.M. (QED):	Long-$-\frac{1}{r}$:	1/137:	U(1):	Photons	
Weak (Y.M):	Short-$\delta(r)$:	10^{-5}:SU(2):		$W^{\pm}; Z^0$	
Strong QCD:	hort- 1Fm	14:	SU(3):	8-gluons	
Gravitation:	Long- $\frac{1}{r}$	1:	SL(2,C)?	Gravitons	... (13)

6. Symmetry Breaking

In some cases the internal symmetries are only approximate and they are so badly broken that these are hardly recognized. To understand why some symmetries are exact and some are approximate, we must look into dynamics. It can be done either on a fundamental level or on the phenomenological level. More we learn about the properties of fundamental interactions at phenomenological level, better we can hope to appreciate how the symmetries are broken, But we can be more ambitious and try to understand the broken symmetry at the fundamental level. One of the way to do it is within the framework of field theory by either constructing a Hamiltonian containing the terms which are no more invariant under the relevant transformationsor by breaking the symmetry spontaneously where we construct the Hamiltonian which has symmetry in question but arrange such that its physical states do not obey this symmetry. Thus we have:

Dynamical Symmetry Breaking (DSB) $\Leftrightarrow$ $Adding\ Massive\ Term\ to\ Hamiltonian$

Spontaneous Symmetry Breaking (SSB): Ground State does not display Symmetry of H.

$$\Downarrow$$

$V(\varphi) = \frac{1}{2}\mu^2|\varphi|^2 + \frac{\lambda}{4}|\varphi|^4$: Global Symmetry $\varphi \to \varphi' = e^{i\theta}\varphi$

For $\mu^2 < 0: V(\varphi)$ is extreme when

$\varphi = 0\ and\ \pm\sqrt{(\frac{-\mu^2}{\lambda})}$

$$\Downarrow$$

SSB

$$\Downarrow$$

$< \varphi >_{\mathbf{0}} = \pm \boldsymbol{v} = \pm\sqrt{(\frac{-\boldsymbol{\mu}^2}{\boldsymbol{\lambda}})}$ Ground State Degeneracy

$$\Downarrow$$

Mass creation $\boldsymbol{m} = \boldsymbol{ev}$ (Higgs Mechanism)(14)

7. Unification Programs

The symmetry or the underlying harmony of Nature could lead to many attempts of unification of fundamental forces shown in the table of equation (13). Starting with the unification of the electricity and magnetism in the form of electromagnetic theory and the attempts of Kaluza and Klein [14] to unify the electromagnetic theory with gravitation, the first successful program was Salam-Weiberg-Glassow Model [15, 16, 17] of the unification of weak interaction [gauge group SU(2)] with the electromagnetic interaction [gauge group U(1)]:

(a) Electroweak (SWG) :

$$SU(2) \times U(1) \quad \dots (15)$$

Next grand step in the direction of unification has been the standard model [15] which unified weak, electromagnetic and strong interactions with gauge symmetries SU(2), U(1) and $SU(3)_c$ respectively;

(b) Standard Model (Georgi and Glashow) [16,17]:

$$SU(3)_c \times SU(2) \times U(1) \qquad \ldots (16)$$

$$\Downarrow$$

Lepton-Hadron Symmetry

$$\Downarrow$$

Conservation of Charge :

$$Q_{had} + Q_{lep} = 0 \qquad \ldots (17)$$

This unification of QCD of $SU(3)_c$ and QFD of electroweak has been one of the greatest triumphs of physics and believed to be free from mathematical in consistencies. It successfully explained the lepton-hadron symmetry and the conservation of the total charge. But it has following several unresolved problems and paradoxes associated with it:

(i) *Too complicated*

(ii) *Contains Many Parameters (18) (3 gauge couplings + 2 CP-violating θ parameters,+nine fermion masses+ three Cabibo mixing angles+1 CP violating Cabbibo phase)*

(iii) *Does not Explain CP-Violation*

(iv) *Does not Explain Quantization of Charge*

(v) *Gravity Not Included*

(vi) *Does not predict fermion mixing angles*

(vii) *Does not explain empirical absence of large cosmological terms*

This last problem is related with big number $D \approx 10^{40}$ which appears in many strange relations between gravitational, cosmological, and quantum atomic quantities. For instance [18]:

(a) Ratio of Coulomb and Newton forces $\approx D \approx 10^{40}$

(b) Ratio of observed meta-galaxy and nuclear dimensions:

$$\frac{R\ (Universe\ radius)}{r(\ proton\ ratio)} \approx D \approx 10^{40}$$

(c) Ratio of Salam's strong gravity and Newton-Einstein gravity constants. In gravity of hadrons (strong gravity) the gravity potential is

$$\emptyset_s = G_s \frac{m}{rc^2} \qquad \ldots (17)$$

while universe potential is $\emptyset = \frac{GM}{Rc^2}$,

where m is the mass of the proton, M is the mass of the universe, r is the proton radius ans R is the radius of the universe. It gives

$$\frac{M}{m} \sim D^2 \approx 10^{80}$$
$$\frac{R}{r} \sim \mathrm{D} \approx 10^{40}$$
$$\frac{G_S}{G} \sim \mathrm{D} \approx 10^{40} \qquad \text{... (18)}$$

Diractried to develop a new cosmology including these big numbers.

In view of these short comings of standard model, it could not be considered as the ultimate theory of unification of fundamental forces.

(c) Grand Unified Theories (GUTs)

Visualizing the consequences of spontaneous symmetry breaking, left-right symmetry and combination of gauge sector and Higg's sector, there came very attractive Grand Unified Theeories (GUT's) [19,20] as the next step in unification programs. GUT's have many attractive features like:

(a) Unified: EM; Weak & Strong Forces with a Single coupling Constant

(b) Explained: (i) Equality of Charges on Proton and Positron

$\Downarrow$

Charge Operator **Q** is Traceless (generator of Gauge Symmetry)

(ii) Dynamical generation of baryon-antibaryon asymmetry of the universe

(CP violation)

(iii) Quantization of Electric Charge

$\Downarrow$

(iv) Contains Monopoles (21- 24) and Dyons (25-39) as Intrinsic Parts

which catalyse baryon number non-conserving processes.

Inspite of the remarkable success of GUT's in answering some unresolved difficulties of Standard Model, there were left some unsettled problems in these theories of unification also. For instance:

(i) Desert Between Weak Interaction Mass Scale M_W and Unification mass Scale M_X

(ii) $\frac{M_W^2}{M_X^2} < 10^{-24}$ (Gauge Hierarchy Problem)

(iii) Gravity Not Included

However some noticeable attempts to include gravity in unification have been made by Kaluza –Klein [14] and Rajput [40].

In order to make an attempt to resolve these difficulties of GUT's, the idea of super grand unification was put forward to unify gravity more successfully with other fundamental forces (no more step motherly treatment to gravity) and consequently notion of ultimate harmony (symmetry) has been visualized in terms of Super-Symmetry (SUSY) [41-46], Super-gravity (Sugra) (47, 48) and Higher Dimensional Space –Time [38-47] incorporating the natural framework of Super-Strings [50-55].

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