STABILITY ANALYSIS OF THE MODEL UNDER INFLUENCE OF HARMFUL ALGAE TO THE GROWING CORAL REEF COMMUNITY

KSHAMA MISHRA

Assistant Professor, Institute of Research Develpoment and Training, Kanpur

RECEIVED : 2 August, 2016

In this paper we are analyzing the effect of harmful algae to coral reef community.

INTRODUCTION

Recent decades have witnessed a dramatic decline in reef growth in almost every region where reefs are found (Wilkinson, [14]). There is an urgent need for better predective tools to increase our understanding of the responses of corals towards changing environment. In this chapter we are discussing the stability of the model under the influence of harmful algae. Algae play an important role in the declination of the coral-reef community. This model has been generated with the help of a metric of a Finsler Space of two-dimension, using the theory of calculus of variations.

Harmful algae badly effect the coral reefs. Due to toxicity of these algae, the growth of coral reef decreases. We are taking three main harmful algae Crustose Coralline Algae, Turf Micro Algae and Frandose Macro Algae [10].

FORMATION OF A FINSLERIAN MODEL

Consider the metric function of a two-dimensional Finsler Space, which is given by

$$L^2 = \frac{1}{2} e^{2\alpha_i x^i (l^2 + 1) + 2l \tan^{-1}(\dot{x}^1 / \dot{x}^2)} \{(\dot{x}^1)^2 + (\dot{x}^2)^2\}, \quad \dots (2.1)$$

where x^1, x^2 are Cartesian co-ordinates on R^2 and l is any non-zero real number.

We shall construct our model from equation (2.1) using the theory of calculus of variations [7]. The Euler-Lagrange equation for the metric function L^2 is given by

$$\frac{d}{ds}\left(\frac{\partial L^2}{\partial \dot{x}^i}\right) - \frac{\partial L^2}{\partial x^i} = 0, \; i = 1, 2 \quad \dots (2.2)$$

where s is an arc length.

For i =1, the equation becomes

$$\frac{d}{ds}\left(\frac{\partial L^2}{\partial \dot{x}^1}\right) - \frac{\partial L^2}{\partial x^1} = 0,$$

or $$\ddot{x}^1 + \dot{x}^1\left[(2\alpha_1\dot{x}^1 + 2\alpha_2\dot{x}^2)(l^2+1) + \frac{2l(\dot{x}^2\ddot{x}^1 - \dot{x}^1\ddot{x}^2)}{(\dot{x}^1)^2 + (\dot{x}^2)^2}\right]$$

$$+l\dot{x}^2\left[2(\alpha_1\dot{x}^1 + \alpha_2\dot{x}^2)(l^2+1) + \frac{2l(\dot{x}^2\ddot{x}^1 - \dot{x}^1\ddot{x}^2)}{(\dot{x}^1)^2 + (\dot{x}^2)^2}\right] + l\ddot{x}^2 - \alpha_1\{(\dot{x}^1)^2 + (\dot{x}^2)^2\} = 0, \quad \ldots (2.3)$$

Similarly for $i = 2$, the Euler-Lagrange equation is given by

$$\ddot{x}^2 + \dot{x}^2\left[2(\alpha_1\dot{x}^1 + \alpha_2\dot{x}^2)(l^2+1) + \frac{2l}{(\dot{x}^1)^2 + (\dot{x}^2)^2}\{\dot{x}^2\ddot{x}^1 - \dot{x}^1\ddot{x}^2\}\right] - l\ddot{x}^1$$

$$-l\dot{x}^1\left\{\begin{array}{l} 2(\alpha_1\dot{x}^1 + \alpha_2\dot{x}^2)(l^2+1) \\ + \dfrac{2l}{(\dot{x}^1)^2 + (\dot{x}^2)^2}\{\dot{x}^2\ddot{x}^1 - \dot{x}^1\ddot{x}^2\} \end{array}\right\} - \{(\dot{x}^1)^2 + (\dot{x}^2)^2\}\alpha_2(l^2+1) = 0 \quad \ldots (2.4)$$

Solving equations (2.3) and (2.4), we get the equations of geodesics [4, 5, 11, 12], which are given by

$$\left.\begin{array}{l} \dfrac{d^2x^1}{ds^2} + (\alpha_1 - l\alpha_2)(\dot{x}^1)^2 + (-\alpha_1 + l\alpha_2)(\dot{x}^2)^2 + 2(\alpha_2 + \alpha_1 l)\dot{x}^1\dot{x}^2 = 0, \\ \dfrac{d^2x^2}{ds^2} + (-\alpha_2 - l\alpha_1)(\dot{x}^1)^2 + (\alpha_2 + l\alpha_1)(\dot{x}^2)^2 + 2(\alpha_1 + \alpha_2 l)\dot{x}^1\dot{x}^2 = 0. \end{array}\right\} \quad \ldots (2.5)$$

Substituting $s = e^{\lambda t}$ and using Volterra-Hamilton Theory [1, 2, 3], we get

$$\left.\begin{array}{l} \dfrac{dN^1}{dt} + (\alpha_1 - l\alpha_2)(N^1)^2 + (-\alpha_1 + l\alpha_2)(N^2)^2 + 2(\alpha_2 + l\alpha_1)N^1N^2 - \lambda N^1 = 0, \\ \dfrac{dN^2}{dt} - (\alpha_2 + l\alpha_1)(N^1)^2 + (\alpha_2 + l\alpha_1)(N^2)^2 + 2(\alpha_1 - l\alpha_2)N^1N^2 - \lambda N^2 = 0. \end{array}\right\} \quad \ldots (2.6)$$

where N^1 denotes the population of coral and N^2 denotes the population of harmful algae. α_1, α_2 and λ are positive constants.

Equilibrium Points [4, 6, 13]

Equilibrium points correspond to constant solutions of the system of differential equations *i.e.*

$$\frac{dN^1}{dt} = 0 \quad \text{and} \quad \frac{dN^2}{dt} = 0. \quad \ldots (2.7)$$

The equation $\dfrac{dN^1}{dt} = 0$ implies

$$-2(\alpha_2 + l\alpha_1)N^1N^2 - (-\alpha_1 + l\alpha_2)(N^2)^2 - (\alpha_1 - l\alpha_2)(N^1)^2 + \lambda N^1 = 0 \quad \ldots (2.8)$$

and the equation $\dfrac{dN^2}{dt} = 0$ implies

$$-2(\alpha_1 - l\alpha_2)N^1N^2 + (\alpha_2 + l\alpha_1)(N^1)^2 - (\alpha_2 + l\alpha_1)(N^2)^2 + \lambda N^2 = 0. \quad \ldots (2.9)$$

Let (N_0^1, N_0^2) be the unique non-zero equilibrium point and k is any positive integer.

Substituting $N_0^2 = kN_0^1$ in the equation (2.9), we get

$$N_0^1 = \frac{\lambda}{(\alpha_1 - l\alpha_2) + (-\alpha_1 + l\alpha_2)k^2 - 2(\alpha_2 + l\alpha_1)k} \quad \ldots (2.10)$$

where $\lambda > 0$

and
$$N_0^2 = \frac{k\lambda}{(\alpha_1 - l\alpha_2) + (-\alpha_1 + l\alpha_2)k^2 - 2(\alpha_2 + l\alpha_1)k} \quad \ldots (2.11)$$

N_0^1 and N_0^2 both are positive if

$$(\alpha_1 - l\alpha_2) + (-\alpha_1 + l\alpha_2)k^2 - 2(\alpha_2 + l\alpha_1)k > 0. \quad \ldots (2.12)$$

For convenience taking $\alpha_1 = \alpha_2 = \alpha$, above condition becomes

$$\alpha(1-l) + \alpha(l-1)k^2 - 2\alpha(1+l)k > 0.$$

LINEARIZATION [6, 13]

The stability of the model predicts the effect of harmful algae to the growing coral reef.

The stability of a linear system is easy, hence we shall linearise our non-linear dynamical system at equilibrium point (N_0^1, N_0^2).

Let $N^1 = (x_1 + N_0^1)$ and $N^2 = (x_2 + N_0^2)$.

Then the new system is

$$\frac{dx_1}{dt} = x_1\left\{-2\alpha(1+l)N_0^2 - 2\alpha N_0^1(1-l) + \lambda\right\} + x_2\left\{-2\alpha(1+l) - 2\alpha(l-1)N_0^2\right\} - 2\alpha(1+l)x_1x_2$$
$$-\alpha(1-l)x_2^2 - \alpha(1-l)x_1^2 - 2\alpha(1+l)N_0^1N_0^2 - \alpha(1-l) - \alpha(1-l)(N_0^1)^2 + \lambda N_0^1 \quad \ldots (3.1)$$

and

$$\frac{dx_2}{dt} = x_1\left\{-2\alpha(1-l)N_0^2 + 2\alpha N_0^1(1+l)\right\} + x_2\left\{-2\alpha(1-l)N_0^1 - 2\alpha(1+l)N_0^2 + \lambda\right\} - 2\alpha(1-l)x_1x_2$$
$$+\alpha(1+l)x_1^2 - \alpha(1+l)x_{21}^2 - 2\alpha(1-l)N_0^1N_0^2 - \alpha(1+l)(N_0^2)^2 + \alpha(1+l)(N_0^1)^2 + \lambda N_0^2$$

The linearised system can be written as

$$\dot{x} = AX \quad \ldots (3.2)$$

where
$$A = \begin{bmatrix} -2\alpha(1+l)N_0^2 - 2\alpha(1-l)N_0^1 + \lambda & -2\alpha(1+l)N_0^1 - 2\alpha(l-1)N_0^2 \\ -2\alpha(1-l)N_0^2 + 2\alpha(1+l)N_0^1 & -2\alpha(1-l)N_0^1 - 2\alpha(1+l)N_0^2 + \lambda \end{bmatrix}$$

and $$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The characteristic equation [8] of the matrix A is

$$|A - \nu I| = 0, \qquad \ldots (3.3)$$

which may be written as

$$[-2\alpha(1+l)\,N_0^2 - 2\alpha N_0^1(1-l) + \lambda - \nu]^2 + [2\alpha(1+l)\,N_0^1 + 2\alpha N_0^2(l-1)]^2 = 0.$$

This implies

$$\nu^2 - 2\nu\{\lambda - (2\alpha\,(1+l)\,N_0^2 + 2\alpha\,(1-l)\,N_0^1)\} + \{\lambda - [2\alpha\,(1+l)\,N_0^2 + 2\alpha\,(1-l)\,N_0^1]\}^2$$
$$+[2\alpha\,(1+l)\,N_0^1 + 2\alpha(1-l)\,N_0^2]^2 = 0. \qquad \ldots (3.4)$$

The equation (3.4) being a quadratic equation gives two roots. These roots will be equal or distinct according to

$$4\left\{\lambda - \left(2\alpha(1+l)N_0^2 + 2\alpha(1-l)N_0^1\right)\right\}^2 - 4\begin{bmatrix}\left\{\lambda - \left(2\alpha(1+l)N_0^2 + 2\alpha(1-l)N_0^1\right)\right\} + \\ \left(2\alpha(1+l)N_0^1 + 2\alpha(l-1)N_0^2\right)^2\end{bmatrix} 0 \text{ or } \neq 0 \qquad \ldots (3.5)$$

If $$4\left\{\lambda - \left(2\alpha(1+l)N_0^2 + 2\alpha(1-l)N_0^1\right)\right\}^2 - 4\begin{bmatrix}\left\{\lambda - \left(2\alpha(1+l)N_0^2 + 2\alpha(1-l)N_0^1\right)\right\} + \\ \left(2\alpha(1+l)N_0^1 + 2\alpha(l-1)N_0^2\right)^2\end{bmatrix} = 0$$

We observe that both the eigen values of the matrix A are negative

$$\lambda < N_0^1\{2\alpha\,(1+l)k + 2\alpha\,(1-l)\}. \qquad \ldots (3.6)$$

Substituting the value of N_0^1 from (2.12), in equation (3.6), we get

$$\alpha\,(l-1)\,k^2 - 4\alpha\,(1+l)\,k - \alpha\,(1-l) < 0.$$

This implies

$$\left(\frac{k^2+1}{k}\right) < \frac{4(l+1)}{(l-1)}. \qquad \ldots (3.7)$$

In view of condition (3.7), all the eigen values of the matrix A are negative, and hence our model is stable [6, 9, 13].

If we take the non-zero condition of equation (3.5), there arise two cases:

$$\left.\begin{array}{l}\text{(i) } 4\left\{\lambda - \left(2\alpha(1+l)N_0^2 + 2\alpha(1-l)N_0^1\right)\right\}^2 - 4\begin{bmatrix}\left\{\lambda - \left(2\alpha(1+l)N_0^2 + 2\alpha(1-l)N_0^1\right)\right\} \\ +\left(2\alpha(1+l)N_0^1 + 2\alpha(l-1)N_0^2\right)^2\end{bmatrix} > 0 \\ \text{and (ii) } 4\left\{\lambda - \left(2\alpha(1+l)N_0^2 + 2\alpha(1-l)N_0^1\right)\right\}^2 - 4\begin{bmatrix}\left\{\lambda - \left(2\alpha(1+l)N_0^2 + 2\alpha(1-l)N_0^1\right)\right\} \\ +\left(2\alpha(1+l)N_0^1 + 2\alpha(1-l)N_0^2\right)^2\end{bmatrix} < 0\end{array}\right\} \ldots (3.8)$$

Case 1. In this case the eigen values are

$$\nu_1 = (-2\alpha(1+l)N_0^2 - 2\alpha(1-l)N_0^1 + \lambda) + \sqrt{\left\{\lambda - \left(2\alpha(l+1)N_0^2 + 2\alpha(l-1)N_0^1\right)\right\}^2 - \left[\left\{\lambda - \left(2\alpha(l+1)N_0^2 + 2\alpha(1-l)N_0^1\right)\right\} + (2\alpha(1+l)N_0^1 + 2\alpha(l-1)N_0^2)\right]}$$

and

$$\nu_2 = (-2\alpha(1+l)N_0^2 - 2\alpha(1-l)N_0^1 + \lambda) - \sqrt{4\left\{\lambda - \left(2\alpha(l+1)N_0^2 + 2\alpha(1-l)N_0^1\right)\right\}^2 - \left[\left\{\lambda - \left(2\alpha(l+1)N_0^2 + 2\alpha(1-l)N_0^1\right)\right\} + (2\alpha(1+l)N_0^1 + 2\alpha(l-1)N_0^2)\right]}$$

The root ν_2 is negative, if

$$2\alpha N_0^1\{(1+l)k + (1-l)\} > \lambda. \qquad \ldots(3.9)$$

The root ν_1 is negative, if

$$2\alpha(1+l)N_0^2 + 2\alpha(1-l)N_0^1 - \lambda > \sqrt{4\left\{\lambda - \left(2\alpha(1+l)N_0^2 + 2\alpha(1-l)N_0^1\right)\right\}^2 - 4\left[\begin{array}{l}\left\{\lambda - \left(2\alpha(1+l)N_0^2 + 2\alpha(1-l)N_0^1\right)\right\} \\ + \left(2\alpha(1+l)N_0^1 + 2\alpha(l-1)N_0^2\right)^2\end{array}\right]} \qquad \ldots (3.10)$$

which implies

$$\lambda^2 - 2\lambda + 4\alpha N_0^1\{(1+l)k + (1-l)\}(1-\lambda) + 4\alpha^2(N_0^1)^2 \times \{((1+l)k + (1-l))^2 - (1+l) + (l-1)k^2\} < 0. \qquad \ldots (3.11)$$

Substituting the value of N_0^1 from equation (2.12) in equation (3.11) we get

$$\lambda\left\{\frac{(k^2(l-1) - 2(l+1)k + 1 - l)^2 + 16(lk+1)(k-l)}{(k^2(l-1) - 2(1+l)k + 1 - l)}\right\} < 2\left\{\begin{array}{l}k^2(l-1) - 2(l+1)k + 1 - l \\ -2((1+l)k + (1-l))(1-\lambda)\end{array}\right\}.$$

This implies

$$\lambda < 2\{k^2(l-1) - 2(1+l)k + 1 - l - 2((1+l)k + (1-l))(1-\lambda)\} \times \left(\frac{k^2(l-1) - 2(1+l)k + 1 - l}{k^2(l-1) - 2(1+l)k + 1 - l)^2 + 16(lk+1)(k-l)}\right) \qquad \ldots (3.12)$$

In this case both the eigen values ν_1 and ν_2 are negative, therefore our model is stable.

Case 2. If

$$4\left\{\lambda - \left(2\alpha(1+l)N_0^2 + 2\alpha(1-l)N_0^1\right)\right\}^2 - 4\left[\begin{array}{l}\left\{\lambda - \left(2\alpha(1+l)N_0^2 + 2\alpha(1-l)N_0^1\right)\right\} \\ + \left(2\alpha(1+l)N_0^1 + 2\alpha(l-1)N_0^2\right)\end{array}\right] < 0,$$

There arise two cases

$$\left.\begin{array}{ll} \text{(i)} & \lambda - \left(2\alpha(1+l)N_0^2 + 2\alpha(1-l)N_0^1\right) = 0 \\ \text{(ii)} & \lambda \neq 2\alpha N_0^1[(1+l)k + (1-l)]. \end{array}\right\} \quad \ldots (3.13)$$

Case (i). If $\lambda = 2\alpha N_0^1\{(1+l)k + (1-l)\}$, then both the eigen values are purely imaginary and hence the model is asymptotically stable.

Case (ii). If $\lambda \neq 2\alpha N_0^1[(1+l)k + (1-l)]$, then both the eigen values are complex conjugate to each other. Then the model is stable if the real part of the root is negative and it is unstable if the real part of the root is positive [5, 13].

Conclusion

We conclude that our Finslerian model (given by the equation (2.6)), constructed with the help of the metric (given by the equation (2.1)) is stable when any one of the conditions (3.6) with (3.7) or (3.8(i)) is satisfied. This model is asymptotically stable when (3.13(i)) holds, our model is unstable when (3.8(ii)) with (3.13(ii)) is satisfied. The stability of the model shows that algae badly effect the growth of the coral reef when the conditions of stability mentioned above are satisfied.

References

1. Antonelli, P.L., A brief introduction to Volterra-Hamilton theory in ecological Mathematical, In proceeding of Int. Workshop on the Population Dynamics of Outbreaks, *Math. Comp. Modeling*, **13(6)**, 19-23(1990).
2. Antonelli, P.L., Finsler Volterra-Hamilton system in Ecology, *Tensor*, N.S. 22-31 (1991).
3. Antonelli, P.L. and Bradbury, R.H., Volterra-Hamilton models in the Ecology and Evolution of colonial organisms, *World Scientific Singapore* (1996).
4. Antonelli, P.L., *Hand book of Finsler Geometry*, Vol. **1**, Kluwer Academic Publishers, Boston (2003).
5. Antonelli, P.L., Hand book of Finsler Geometry, Vol. **2**, Kluwer Academic Publishers, Boston (2003).
6. Belinda, Barnes and Fulford, Glenn R., Mathematical modeling with case studies, London and New York (2002).
7. Gupta, A.S., Calculus of Variation with Applications, Prentice-Hall of India Private Limited, New Delhi, (2003).
8. Hoffman Kenneth and Kunz Ray, Linear Algebra, Prentice Hall of India Private Limited, New Delhi (1998).
9. Kapur, J.N., Mathematical Modeling in Biology and Medicine, East-West Press Private Limited, New Delhi (2000).
10. Littler Mark, M. Littler, D.S. and Brooks Barett, L., Harmful Algae on tropical coral reefs, Bottom-up eutrophication and Top-down herbivory, Harmful Algae (2005).
11. Matsumoto, M., Geodesics of two dimensional Finsler spaces, *Math. Comput. Modeling*, **20**, 1-23 (1994).
12. Matsumoto, M., Foundations of Finsler Geometry and Special Finsler spaces, Kaiseisha Press, Saikawa, Otsu, Japan, (1986).
13. Perko, L., Differential equations and Dynamical systems, Springer, NewYork (1996).
14. Wilkinson, C., Status of Coral Reef of the world, Australia Institute of Marine Science, *Townsville*, 378 pp (2002).