ON SOME LABELINGS OF DIRECTED PATHS

D. PRATHAP

Department of Mathematics, CMR Institute of Technology, Bangalore

AND

T.V. PRADEEP KUMAR

Department of Mathematics, ANU College of Engineering & Technology, Acharya Nagarjuna University

RECEIVED : 19 May, 2016

J. Baskar Babujee and N. Prabhakara Rao [2] introduced the concept of (1, 1) indegree vertex magic and (1, 1) out degree vertex magic labeling and studied for different types of graphs. In [1], the concept of (1, 1) vertex bimagic labeling was introduced and studied for the disjoint union of n copies of P_2 the nP_2 graph. Inspired by the above two papers the results of this paper on directed paths are obtained. Different types of indegree and outdegree vertex magic, antimagic and bimagic labelings are obtained for paths and for a class of their disconnected union.

KEYWORDS : Indegree, outdegree, magic, antimagic, bimagic, labelling.

AMS Subject Classification: 05C78.

Introduction

In 1967 Alex Rosa [11] used graph labelling as a tool to attack the problem of cylindrically decomposing the complete graphs into trees. Since then several papers have been published and studied [7]. This rapid growth in the study of graph labelings is not only due its mathematical importance but also because of wide range of applications. Graph labelings are widely used in the study of X-rays, cryptography, astronomy, coding theory, circuit design etc., Directed graphs, in particular, have applications in many physical and scientific situations such as flow networks with valves in pipes, electric networks etc., Most part of these studies focused on finding different types of labelings for particular classes of graphs. In this paper, we consider directed paths and find labeling for them and for their disconnected union. Since the graphs under consideration are directed more emphasis is given to the vertices and the sums of the labels with respect to the vertices. The labelings that we study are indegree vertex antimagic and outdegree vertex antimagic, indegree vertex magic and outdegree vertex magic as well as indegree vertex bimagic and outdegree vertex bimagic for directed paths and for a class of disconnected union of paths. J. Baskar Babujee and N. Prabhakara Rao [2] introduced the concept of (1, 1) indegree vertex magic and (1, 1) out degree vertex magic labeling and studied for different types of graphs.

In this paper, we find the number of ways in which these labelings can be given is also obtained.

In [4], the concept of (1, 1) indegree vertex magic and (1, 1) outdegree vertex magic labeling was introduced and studied for different types of graphs. In [1], the concept of (1, 1) vertex bimagic labeling was introduced and studied for the disjoint union of n copies of P_2 the nP_2 graph. Inspired by the above two papers the results of this paper on directed paths are obtained. Here the notation (1, 0) or (1, 1) should not be considered as the one defined in [12]. (1, 0) labelling in this paper is referred to as the one defined for a graph whose vertices alone are labelled and (1, 1) labelling is referred to as the one for graphs whose vertices and edges are labelled.

The graphs under consideration are finite, simple and directed paths. Vertices of P_n, $n \geq 2$ are denoted by v_i, $1 \leq i \leq n$ and the edge to the right of v_i is denoted by e_i. m copies of paths P_n is denoted by mPn. The following are defined to describe the paths and labelings of this paper.

1.1. Definition [9] : (i) A **graph** $G = (V, E)$ consists of a set of objects $V = \{v_1, v_2, ...\}$ called vertices, and another set $E = \{e_1, e_2, ...\}$ whose elements are called edges, such that each edge is identified with an unordered pair (v_i , v_j) of vertices. (ii) The vertices v_i, v_j associated with edge e_k are called end vertices of e_k.

The most common representation of a **graph** is by means of a diagram, in which the vertices are represented as points and edges as line segments or arcs joining its end vertices. Most of the time the diagram itself is referred to as the **graph**. If each edge (v_i , v_j) of G is considered as an ordered pair then the graph is called a **directed graph.**

1.1. Definition [9] : A **walk** is defined as a finite alternating sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it.

1.2. Definition [9] : No edge appears more than once in a walk. A vertex, however, may appear more than once in a walk. It is possible for a walk to begin and end at the same vertex. Such a walk is called a **closed walk**. A walk that is not closed is called an **open walk**.

1.3. Definition [9] : An open walk in which no vertex appears more than once is called a **path**.

1.4. Definition [9] : The number of edges in a path is called the **length of a path.**

1.5. Definition [9] : The number of edges incident out of a vertex v_i, is called the **outdegree** of v_i, and is denoted by $d^+(v_i)$.

1.6. Definition [9] : The number of edges incident into of a vertex v_i, is called the **indegree** of v_i, and is denoted by $d^-(v_i)$.

1.7. Definition [2] : A directed graph $G = G(V, E)$ with p vertices and q edges is said to be **(1, 1) indegree vertex antimagic** if there is a bijective function $f : V(G) \cup E(G) \to \{1, 2, ..., p + q\}$ if for each $u \in V$, $f(u) + \sum_{e} f(e) = k$ where k takes different values for different $u \in V$ and for all $e = (v, u) \in E$, $v \in V$. Here the bijective function f is called **(1, 1) indegree vertex antimagic labeling or indegree vertex antimagic total labeling** on G.

1.8. Definition [2] : A directed graph $G = G(V, E)$ with p vertices and q edges is said to be **(1, 1) outdegree vertex antimagic** with a common vertex count k if there is a bijective function $f : V(G) \cup E(G) \to \{1, 2, ..., p + q\}$ if for each $u \in V$, $f(u) + \sum_{e} f(e) = k$ where k takes different values for different $u \in V$ and for all $e = (u, v) \in E$, $v \in V$. Here the bijective function f is called **(1, 1) outdegree vertex antimagic labeling or outdegree vertex antimagic total labelling** on G.

(1, 0) indegree vertex antimagic and (1, 0) outdegree vertex antimagic labelling are defined similary and for a directed graph whose vertices alone are labelled the value of k in the above two definitions is given by $f(u)$ only, where $f : V(G) \to \{1, 2, ..., p\}$ is a bijective function which gives different values for different $u \in V$.

1.9. Definition [2] : A directed graph $G = G(V, E)$ with p vertices and q edges is said to be **(1, 1) indegree vertex magic** with a common vertex count k if there is a bijective function $f : V(G) \cup E(G) \to \{1, 2, ..., p + q\}$ if for each $u \in V$, $f(u) + \sum_e f(e) = k$ for all $e = (v, u) \in E$, $v \in V$. Here the bijective function f is called **(1, 1) indegree vertex magic labeling or indegree vertex magic total labeling** on G.

1.10. Definition [2]: A directed graph $G = G(V, E)$ with p vertices and q edges is said to be **(1, 1) outdegree vertex magic** with a common vertex count k if there is a bijective function $f : V(G) \cup E(G) \to \{1, 2, ..., p + q\}$ if for each $u \in V$, $f(u) + \sum_e f(e) = k$ for all $e = (u, v) \in E$, $v \in V$. Here the bijective function f is called **(1, 1) outdegree vertex magic labeling or outdegree vertex magic total labelling** on G.

1.11. Definition[10] : A directed graph $G = G(V, E)$ with p vertices and q edges is said to be (1, 1) indegree vertex bimagic with two vertex counts k_1 and k_2 if there is a bijective function $f : V(G) \cup E(G) \to \{1, 2, ..., p + q\}$ if for each $u \in V$, $f(u) + \sum_e f(e) = k_1 \; or \; k_2$ for all $e = (v, u) \in E$ with $v \in V$. Here the bijective function f is called **(1, 1) indegree vertex bimagic or indegree vertex bimagic total labeling** on G.

1.12. Definition [10] : A directed graph $G = G(V, E)$ with p vertices and q edges is said to be **(1, 1) outdegree vertex bimagic** with two vertex counts k_1 and k_2 if there is a bijective function $f : V(G) \cup E(G) \to \{1, 2, ..., p + q\}$ if for each $u \in V$, $f(u) + \sum_e f(e) = k_1 \text{ or } k_2$ for all $e = (u, v) \in E$ with $v \in V$. Here the bijective function f is called **(1, 1) outdegree vertex bimagic or outdegree vertex bimagic total labeling** on G.

LABELINGS FOR DIRECTED PATHS

Let us now discuss about the existence of above labeling for directed paths. First it is observed that directed paths are (1, 0) indegree vertex antimagic and (1, 0) outdegree vertex antimagic. Then three indegree vertex magic total and three outdegree vertex magic total labeling are obtained.

It is also proved that there exist five outdegree vertex magic total labeling and five indegree vertex bimagic total labeling for them. Later it is proved that the directed paths $2P_n$ have two outdegree vertex bimagic total labeling and two indegree vertex bimagic total labeling, two outdegree vertex antimagic total labeling and two indegree vertex antimagic total labeling and two outdegree vertex magic total labeling and two indegree vertex magic total labeling.

The main idea is to find the nature of labeling for the directed paths P_n. Since the paths under consideration are directed, as mentioned earlier, it is more appropriate and important to study indegree and outdegree vertex labelings. Initially, different types of indegree and outdegree vertex labelings are identified for directed paths P_n and then for two disjoint copies of P_n.

Let us first label only vertices of P_n. For any labeling f: $\{v_1, v_2, ..., v_n\} \rightarrow \{1, 2, ..., n\}$, it is obvious that directed paths are indegree vertex antimagic as well as outdegree vertex antimagic for the simple reason that the edges are not labeled. Therefore we have

Lemma 2.1 : Directed paths are (1, 0) indegree vertex antimagic and (1, 0) outdegree vertex antimagic.

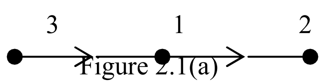


Figure 2.1(a)

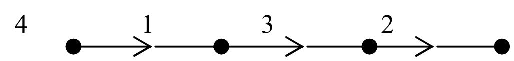


Figure 2.1(b)

So let us now consider paths whose vertices and edges are labeled and examine different types of indegree and outdegree vertex labeling for them.

Define a bijective function $g_1 : V \cup E \rightarrow \{1, 2, ..., 2n - 1\}$ by

$$g_1(v_i) = i, \ 1 \le i \le n$$

$$g_1(e_i) = 2n - 1 - (i - 1), \ 1 \le i \le n - 1$$

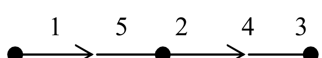


Figure 2.2(a)

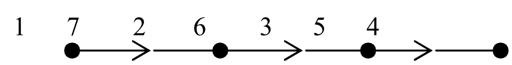


Figure 2.2(b)

It can easily be observed that for any vertex v_i, $1 \le i \le n - 1$,

$$g_1(v_i) + g_1(e_i) = i + 2n - 1 - (i - 1) = 2n.$$

Further, for $1 \le i \le n - 1$,

$$g_1(e_i) + g_1(v_{i+1}) = 2n - 1 - (i - 1) + i + 1 = 2n + 1.$$

Clearly g_1 is an outdegree vertex magic total labeling and an indegree vertex magic total labeling for the paths P_n. Therefore directed paths or directed path graphs P_n are indegree vertex magic and outdegree vertex magic. Thus we have

Lemma 2.2 : Directed paths P_n are indegree vertex magic and outdegree vertex magic.

It is interesting to note that there exist some more indegree vertex magic and outdegree vertex magic labeling for the directed paths. They are identified and described below. They are denoted by g_2, g_3 and are defined as follows. Let us first define $g_2 : V \cup E \to \{1, 2, ..., 2n - 1\}$, by

$$g_2(v_i) = 2n - 1 - (i - 1),\ 1 \le i \le n, \quad \text{and}\ g_2(e_i) = i,\ 1 \le i \le n - 1,$$

Figure 2.3(a)
Figure 2.3(b)

Here again, it can easily be observed that for any vertex v_i, $1 \le i \le n - 1$,

$$g_2(v_i) + g_2(e_i) = 2n - 1 - (i - 1) + i = 2n.$$

Further, for $1 \le i \le n - 1$,

$$g_2(e_i) + g_2(v_{i+1}) = i + 2n - 1 - (i + 1 - 1) = 2n - 1.$$

It can be noted that the labeling given to P_n according to g_2 as shown in Figure 2.3(a) and Figure 2.3(b) can be obtained from the labeling according to g_1 as shown in Figure 2.2(a) and Figure 2.2(b) by an interchange of the labels of v_i and e_i for every $1 \le i \le n - 1$.

We now define $g_3 : V \cup E \to \{1, 2, ..., 2n - 1\}$ by

$$g_3(v_i) = 2i - 1,\ 1 \le i \le n,$$

$$g_3(e_i) = 2(n - 1) - (i - 1)^2,\ 1 \le i \le n - 1,$$

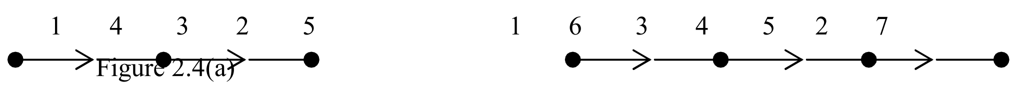


Figure 2.4(a)
Figure 2.4(b)

Here also it is observed that for any vertex v_i, $1 \le i \le n - 1$,

$$g_3(v_i) + g_3(e_i) = 2i - 1 + 2(n - 1) - (i - 1)^2 = 2n - 1.$$

Further, for $1 \le i \le n - 1$, $g_3(e_i) + g_3(v_{i+1}) = 2(n - 1) - (i - 1)^2 + 2(i + 1) - 1 = 2n + 1$.

We now can define g_4 from g_3 by an interchange of labels of v_i and e_i for every $1 \le i \le n - 1$ expecting that it also gives an indegree vertex magic total labeling and an outdegree vertex magic total labeling for the directed paths P_n. However, g_4 slightly deviates from the phenomenon observed above between g_1 and g_2.

Let us define $\quad g_4 : V \cup E \to \{1, 2, ..., 2n - 1\}$ by

$$g_4(v_i) = 2(n - 1) - (i - 1)^2,\ 1 \le i \le n - 1,$$

$$g_4(e_i) = 2i - 1,\ 1 \le i \le n - 1 \text{ and } g_4(v_n) = 2n - 1$$

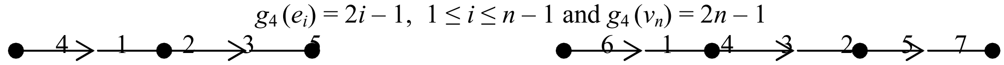


Figure 2.5(a) Figure 2.5(b)

It can easily be observed that for any vertex v_i, $1 \le i \le n - 1$,

$$g_4(v_i) + g_4(e_i) = 2(n - 1) - (i - 1)^2 + 2i - 1 = 2n - 1.$$

Further, for $1 \le i \le n - 2$,

$$g_4(e_i) + g_4(v_{i+1}) = 2i - 1 + 2(n - 1) - (i + 1 - 1)^2 = 2n - 3.$$

and $\quad g_4(e_{n-1}) + g_4(v_n) = 2(n - 1) - 1 + 2n - 1 = 4n - 4.$

The above evaluations imply that though g_4 gives an outdegree vertex magic total labeling for the directed paths it does not give an indegree vertex magic total labeling for the directed paths P_n. When it comes to the sum $g_4\,(e_i) + g_4\,(v_{i+1})$ except for $i = n - 1$ for all other i, $1 \le i \le n - 2$, we get the same sum. Therefore g_4 gives an outdegree vertex magic total labeling for the directed paths but it gives an indegree vertex bimagic total labeling for the directed paths P_n. So only g_1, g_2, g_3 give us indegree vertex magic total labeling and outdegree vertex magic total labeling for the directed paths P_n. If the direction is reversed for the above P_n, still g_1, g_2, g_3 will be outdegree vertex magic total labeling and indegree vertex magic total labeling. However, the indegree vertex magic total and the outdegree vertex magic total get exchanged. So we have

Theorem 2.1 : Directed paths have three indegree vertex magic total labeling and three outdegree vertex magic total labelings.

Since it is observed that g_4 gave an indegree vertex bimagic labeling for P_n, the search for some similar vertex bimagic labeling for P_n is carried out and it gave successful results. In fact four more such bimagic labeling are obtained and they are given below.

Define $g_5 : V \cup E \to \{1, 2, ..., 2n - 1\}$ by

$$g_5\,(v_i) = 2\,(n - 1) - (i - 1)^2,\ \ 1 \le i \le n - 1,$$

$$g_5\,(e_i) = 2i + 1,\ \ 1 \le i \le n - 1 \text{ and } g_5\,(v_n) = 1$$

Figure 2.6(a) Figure 2.6(b)

It can easily be observed that for any vertex v_i, $1 \le i \le n - 1$,

$$g_5\,(v_i) + g_5\,(e_i) = 2\,(n - 1) - (i - 1)^2 + 2i + 1 = 2n + 1.$$

Further, for $1 \le i \le n - 2$, $g_5\,(e_i) + g_5\,(v_{i+1}) = 2i + 1 + 2\,(n - 1) - (i + 1 - 1)^2 = 2n - 1$.

and $$g_5\,(e_{n-1}) + g_5\,(v_n) = 2\,(n - 1) + 1 + 1 = 2n.$$

Therefore, g_5 gives an outdegree vertex magic total labeling and an indegree vertex bimagic total labeling for the directed paths P_n.

Define $g_6 : V \cup E \to \{1, 2, ..., 2n - 1\}$ by

$$g_6\,(v_i) = 2n - 3 - (i - 1)^2,\ \ 1 \le i \le n - 1,$$

$$g_6\,(e_i) = 2i,\ \ 1 \le i \le n - 1 \text{ and } g_6\,(v_n) = 2n - 1$$

Figure 2.7(a)
Figure 2.7(b)

It can easily be observed that for any vertex v_i, $1 \le i \le n - 1$,

$$g_6\,(v_i) + g_6\,(e_i) = 2n - 3 - (i - 1)^2 + 2i = 2n - 1.$$

Further, for $1 \le i \le n - 2$, $g_6\,(e_i) + g_6\,(v_{i+1}) = 2i + 2n - 3 - (i + 1 - 1)^2 = 2n - 3$.

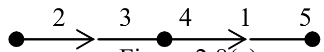
and $$g_6\,(e_{n-1}) + g_6\,(v_n) = 2\,(n - 1) + 2n - 1 = 4n - 3.$$

Therefore, g_6 gives an outdegree vertex magic total labeling and an indegree vertex bimagic total labeling for the directed paths P_n.

Define $g_7 : V \cup E \to \{1, 2, ..., 2n - 1\}$ by

$$g_7\,(v_i) = 2i,\ 1 \le i \le n - 1,$$

$$g_7(e_i) = 2n - 3 - (i-1)^2,\ \ 1 \le i \le n-1,\ g_7(v_n) = 2n - 1$$



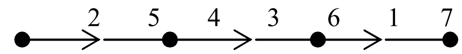


Figure 2.8(a)

Figure 2.8(b)

It can easily be observed that for any vertex v_i, $1 \le i \le n-1$,

$$g_7(v_i) + g_7(e_i) = 2i + 2n - 3 - (i-1)^2 = 2n - 1.$$

Further, for $1 \le i \le n-2$, $g_7(e_i) + g_7(v_{i+1}) = 2n - 3 - (i-1)^2 + 2(i+1) = 2n + 1.$

aqnd $$g_7(e_{n-1}) + g_7(v_n) = 2n - 3 - (n-1-1)^2 + 2n - 1 = 2n.$$

Therefore, g_7 gives an outdegree vertex magic total labeling and an indegree vertex bimagic total labeling for the directed paths P_n.

Define $g_8 :\ V \cup E \to \{1, 2, ..., 2n-1\}$ by

$$g_8(v_i) = 2i + 1,\ 1 \le i \le n-1,$$

$$g_8(e_i) = 2n - 2 - (i-1)^2,\ \ 1 \le i \le n-1,\ g_8(v_n) = 1$$

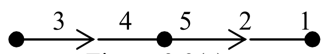


Figure 2.9(a)

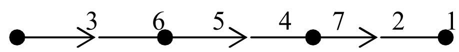


Figure 2.9(b)

It can easily be observed that for any vertex v_i, $1 \le i \le n-1$,

$$g_8(v_i) + g_8(e_i) = 2i + 1 + 2n - 2 - (i-1)^2 = 2n + 1.$$

Further, for $1 \le i \le n-2$, $g_8(e_i) + g_8(v_{i+1}) = 2n - 2 - (i-1)^2 + 2(i+1) + 1 = 2n + 3.$

and $$g_8(e_{n-1}) + g_8(v_n) = 2n - 2 - (n-1-1)^2 + 1 = 3$$

Therefore, g_8 gives an outdegree vertex magic total labeling and an indegree vertex bimagic total labeling for the directed paths P_n.

Here it can be observed that when direction of the paths is reversed, each of the five above bijections g_4, g_5, g_6, g_7 and g_8, which are obtained as outdegree vertex magic total and indegree vertex bimagic labeling for P_n, can be viewed as outdegree vertex bimagic total and indegree vertex magic total labeling for P_n

Further it can also be observed that when the labels of vertices v_i and edges e_i for $1 \le i \le n-1$ are interchanged g_4 gives g_3, g_5 gives g_8, g_6 gives g_7. For this interchange, it is interesting to note that the outdegree vertex magic total labeling and the indegree vertex bimagic total labeling g_4 turns out to be the outdegree vertex magic total and indegree vertex magic total labeling g_3, whereas for the others, that is, for the pairs g_5 & g_8 and g_6 & g_7 the nature of the labeling is not disturbed for such an interchange of labeling.

The five bijective functions g_4, g_5, g_6, g_7 and g_8 given above establish that

Theorem 2.2: Directed paths have five outdegree vertex magic total labeling and five indegree vertex bimagic total labeling.

Alternatively, the theorem can also be stated as "Directed paths have five indegree vertex magic total labeling and five outdegree vertex bimagic total labeling."

Labelings for Disconnected Union of Paths

Considerable work has been done on the disconnected union of path graphs also. Yegnanarayanan [13] proved that nP_3 are edge magic total, when n is odd. *R*. Figueroa-

Centeno, R. Ichishima and F. Muntaner-Batle, [4, 5, 6] proved that nP_i is edge magic total if n is odd and $i = 3, 4, 5$., $2P_n$ is super edge magic if and only if n is not 2 or 3. They also proved that $2P_n$ is super edge magic if and only if n is not 2 or 3and that $2P_{4n}$ is super edge magic for all n. It is also proved [3] that kP_2 is super edge magic if and only if k is odd. We first obtain outdegree vertex bimagic total and indegree vertex bimagic total labeling for the graphs $2P_n$. Here we consider two disjoint copies of the directed path graphs P_n and seek to find the nature of labeling available for them.

Define $h_1 : V \cup E \to \{1, 2, ..., 4n - 2\}$ by

$$h_1(v_i) = i, 1 \le i \le 2n, \quad h_1(e_i) = 4n - 2 - (i - 1), 1 \le i \le n - 1,$$

$$h_1(e_i) = 4n - 2 - (n - 1) - (i - n), n \le i \le 2n - 2.$$

Figure 3.1(a) Figure 3.1(b)

So that we have

$$h_1(v_i) + h_1(e_i) = i + 4n - 2 - (i - 1) = 4n - 1, \qquad 1 \le i \le n - 1$$

and, $$h_1(v_{i+1}) + h_1(e_i) = i + 1 + 4n - 1 - i = 4n, \qquad n \le i \le 2n - 2$$

Further we observe that

$$h_1(e_i) + h_1(v_{i+1}) = 4n - 2 - (i - 1) + i + 1 = 4n, \quad 1 \le i \le n - 1$$

and, $$h_1(e_i) + h_1(v_{i+2}) = 4n - 1 - i + i + 2 = 4n + 1, \qquad n \le i \le 2n - 2$$

This concludes that h_1 is an outdegree vertex bimagic total labeling and an indegree vertex bimagic total labeling for $2P_n$.

Define $h_2 : V \cup E \to \{1, 2, ..., 4n - 2\}$ by

$$h_2(v_i) = 4n - 2 - (i - 1), \qquad 1 \le i \le 2n,$$

$$h_2(e_i) = i, \qquad 1 \le i \le 2n - 2,$$

Figure 3.2(a) Figure 3.2(b)

So that we have

$$h_2(v_i) + h_2(e_i) = 4n - 2 - (i - 1) + i = 4n - 1, \qquad 1 \le i \le n - 1$$

and, $$h_2(v_{i+1}) + h_2(e_i) = 4n - 2 - (i + 1 - 1) + i = 4n - 2, \quad n \le i \le 2n - 2$$

Further we observe that

$$h_2(e_i) + h_2(v_{i+1}) = i + 4n - 2 - (i + 1 - 1) = 4n - 2, \quad 1 \le i \le n - 1$$

and, $$h_2(e_i) + h_2(v_{i+2}) = i + 4n - 2 - (i + 2 - 1) = 4n - 3, \quad n \le i \le 2n - 2$$

This concludes that h_2 is an outdegree vertex bimagic total labeling and an indegree vertex bimagic total labeling for $2P_n$. These two bijective functions conclude that

Theorem 3.1: Directed paths $2P_n$ have two outdegree vertex bimagic total labeling and two indegree vertex bimagic total labeling.

It is interesting to observe that $2P_n$ are outdegree vertex antimagic and indegree vertex antimagic. The following bijections will describe the corresponding labeling.

Define φ_1: $V \cup E \to \{1, 2, ..., 4n - 2\}$ by

$$\varphi_1(v_i) = i, \qquad 1 \le i \le 2n,$$
$$\varphi_1(e_i) = 4n - 2 - (i - 1)^2, \qquad 1 \le i \le n - 1,$$
$$\varphi_1(e_i) = 4n - 3 - (i - n)^2, \qquad n \le i \le 2n - 2.$$

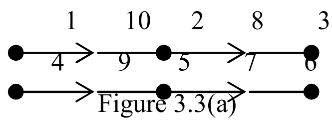


Figure 3.3(a)

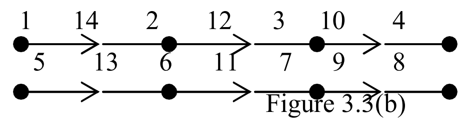


Figure 3.3(b)

So that we have $\varphi_1(v_i) + \varphi_1(e_i) = i + 4n - 2 - (i - 1)^2 = 4n - i$, $1 \le i \le n - 1$

and, $\varphi_1(v_{i+1}) + \varphi_1(e_i) = i + 1 + 4n - 3 - (i - n)^2 = 6n - i - 2$, $n \le i \le 2n - 2$

Further we observe that

$$\varphi_1(e_i) + \varphi_1(v_{i+1}) = 4n - 2 - (i - 1)^2 + i + 1 = 4n - i + 1, \quad 1 \le i \le n - 1$$

and, $\varphi_1(e_i) + \varphi_1(v_{i+2}) = 4n - 3 - (i - n)^2 + i + 2 = 6n - i - 1$, $n \le i \le 2n - 2$

This concludes that φ_1 is an outdegree vertex antimagic total labeling and an indegree vertex antimagic total labeling for $2P_n$

Define φ_2 : $V \cup E \to \{1, 2, ..., 4n - 2\}$ by

$$\varphi_2(v_i) = 4n - 2 - (i - 1), \quad 1 \le i \le 2n,$$
$$\varphi_2(e_i) = 2i - 1, \qquad 1 \le i \le n - 1,$$
$$\varphi_2(e_i) = 2 + (i - n)^2, \qquad n \le i \le 2n - 2.$$

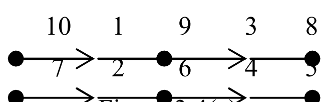


Figure 3.4(a)

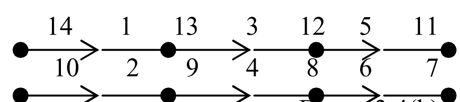


Figure 3.4(b)

So that we have $\varphi_2(v_i) + \varphi_2(e_i) = 4n - 2 - (i - 1) + 2i - 1 = 4n + i - 2$, $1 \le i \le n - 1$

and, $\varphi_2(v_{i+1}) + \varphi_2(e_i) = 4n - 2 - (i + 1 - 1) + 2 + (i - n)^2 = 2n + i$, $n \le i \le 2n - 2$

Further we observe that

$\varphi_2(e_i) + \varphi_2(v_{i+1}) = 2i - 1 + 4n - 2 - (i + 1 - 1) = 4n + i - 3$, $\quad 1 \le i \le n - 1$

and, $\varphi_2(e_i) + \varphi_2(v_{i+2}) = 2 + (i - n)^2 + 4n - 2 - (I + 2 - 1) = 2n + i - 1$, $n \le i \le 2n - 2$

This concludes that φ_2 is an outdegree vertex antimagic total labeling and an indegree vertex antimagic total labeling for $2P_n$. Therefore we have

Theorem 3.2 : Directed paths $2P_n$ have two outdegree vertex antimagic total labeling and two indegree vertex antimagic total labeling.

We now define the following functions which give indegree vertex magic total and outdegree vertex magic total labelings for the directed disconnected paths $2P_n$

Define ψ_1 : $V \cup E \to \{1, 2, ..., 4n - 2\}$ by

$$\psi_1(v_i) = 2i - 1, \qquad 1 \le i \le n,$$
$$\psi_1(v_i) = 2(i - n), \qquad n + 1 \le i \le 2n,$$

$$\psi_1(e_i) = 4n - 2 - (i-1)^2, \quad 1 \le i \le n-1,$$
$$\psi_1(e_i) = 4n - 3 - (i-n)^2, \quad n \le i \le 2n-2.$$

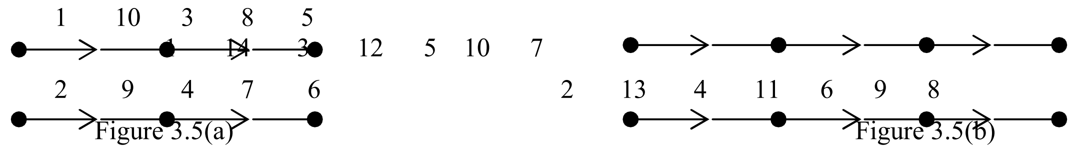


Figure 3.5(a) Figure 3.5(b)

So that we have

$$\psi_1(v_i) + \psi_1(e_i) = 2i - 1 + 4n - 2 - (i-1)^2$$
$$= 4n - 1, \; 1 \le i \le n-1$$

and, $$\psi_1(v_{i+1}) + \psi_1(e_i) = 2(i + 1 - n) + 4n - 3 - (i-n)^2$$
$$= 4n - 1, \; n \le i \le 2n-2$$

Further, we observe that

$$\psi_1(e_i) + \psi_1(v_{i+1}) = 4n - 2 - (i-1)^2 + 2(i+1) - 1 = 4n + 1, \; 1 \le i \le n-1$$

and, $$\psi_1(e_i) + \psi_1(v_{i+2}) = 4n - 3 - (i-n)^2 + 2(i + 2 - n) = 4n + 1, \; n \le i \le 2n-2$$

Define $\psi_2 : V \cup E \to \{1, 2, ..., 4n-2\}$ by

$$\psi_2(v_i) = 4n - 2 - (i-1)^2, \quad 1 \le i \le n,$$
$$\psi_2(v_i) = 4n - 1 - (i-n)^2, \quad n+1 \le i \le 2n,$$
$$\psi_2(e_i) = 2i - 1, \quad 1 \le i \le n-1,$$
$$\psi_2(e_i) = 2(i - (n-1)), \quad n \le i \le 2n-2.$$

Figure 3.6(a) Figure 3.6(b)

So that we have $\psi_2(v_i) + \psi_2(e_i) = 4n - 2 - (i-1)^2 + 2i - 1 = 4n - 1, \; 1 \le i \le n-1$

and, $$\psi_2(v_{i+1}) + \psi_2(e_i) = 4n - 1 - (i + 1 - n)^2 + 2(i - (n-1)) = 4n - 1, \; n \le i \le 2n-2$$

Further, we observe that

$$\psi_2(e_i) + \psi_2(v_{i+1}) = 2i - 1 + 4n - 2 - (i + 1 - 1)^2 = 4n - 3, \quad 1 \le i \le n-1$$

and, $$\psi_2(e_i) + \psi_2(v_{i+2}) = 2(i - (n-1)) + 4n - 1 - (i + 2 - n)^2 = 4n - 3, \; n \le i \le 2n-2$$

The two bijective functions yield the following

Theorem 3.3 : Directed paths $2P_n$ have two outdegree vertex magic total labeling and two indegree vertex magic total labelling.

References

1. Baskar, Babujee J., Bimagic labeling in path graphs, *The Mathematics Education*, Vol. **38**, No. **1**, 12-16 March (2004).
2. Baskar, Babujee J. and Rao, N. Prabhakara, Labelings in digraph, *The Mathematics Education*, Vol. **39**, No. **2**, , 73-78 June (2005).
3. Chen, Z., On super edge magic graphs, *J. Combin Math. Combin. Comput*., **38**, 55-64 (2001).
4. Figueroa-Centeno, R., Ichishima, R. and Muntaner-Batle, F., The place of super edge magic labelings among other classes of labeling, *Discrete Mathematica*, **23**, 153-168 (2001).

5. Figueroa-Centeno, R., Ichishima, R. and Muntaner-Batle, F., On Super edge magic graphs, *Ars Combin*., **64**, 81-95 (2002).
6. Figueroa-Centeno, R., Ichishima, R. and Muntaner-Batle, F., On edge magic labeling of certain disjoint unions of graphs, *Australasian Journal of Combinatorics*, **32**, 225-242 (2005).
7. Gallian, J.A., A dynamic survey of graph labeling, 11^{th} edition, *The Electronic Journal of Combinatorics*, **15**, #DS6 (2008).
8. Kotzig, A. and Rosa, A., Magic Valuations of finite graphs, *Canadian Mathematical Bulletin*, **13**, 451–461 (1970).
9. Narsing, Deo, Graph Theory with applications to engineering and computer science, Prentice Hall of India, Eastern Economy Edition, 25^{th} Printing, June (2003).
10. Prathap, D., On some magic and bimagic labeling of disconnected cycles, *Acta Ciencia Indica*, Vol. **XXXVI** M, No. **4**, 585 -588 (2010).
11. Rosa, A., On certain valuations of the vertices of a graph, Theory of Graphs (Internat. Symposium, Rome, July 1966) Gordon and Breach, N.Y. and Dunod Paris, 349-355 (1967).
12. Sugeng, K.A. and Miller, M., Relationship between adjacency matrices and super (*a, d*) edge antimagic total labeling, *J. Combin. Math.Combin. Computing*, **55**, 71-82 (2005).
13. Yegnanarayanan, V., On Magic graphs, *Util. Math*., **59**, 181–204 (2001).

❑