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EDGE MINIMAL DOMINATING GRAPH

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In this paper, we introduce a new class of graph is known as edge minimal dominating graph $Ed(G)$ of a graph G. Also we obtain the basic properties like order, size, girth, vertex and edge connectivity, covering invariants of $Ed(G)$. Further we obtain those graphs whose $Ed(G)$ is complete bipartite, k-trees and eulerian.

KEYWORDS : Dominating set, minimal dominating set, domination number, upper domination number.

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INTRODUCTION

The graph considered here are finite, undirected without loops or multiple edges. Any undefined term in this paper may be found in Harary [1].

Let $G = (V, E)$ be a graph. A set $D \subseteq V$ is called a dominating set if every vertex $v \in V$ is either an element of D or is adjacent to an element of D. A dominating set D is a minimal dominating set if no proper subset $D' \subset D$ is a dominating set. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set in G. The upper domination number $\Gamma(G)$ of G is the maximum cardinality of a minimal dominating set in G. The girth of a graph G, denoted by $g(G)$, is the length of a shortest cycle (if any) in G. Note that this term is undefined if G has no cycles.

The simplest way to define a k-tree for $k \geq 1$ is by recursion. A k-tree of order $k+1$ is a complete graph of order $k+1$. A k-tree of order $p+1$, $p \geq k+1$, can be obtained by joining a new vertex to any k mutually adjacent vertices of k-tree of order p. Let us state some known facts on k-trees:

(i) A k-tree of order $p \geq k+1$ is k-connected.

(ii) A k-tree of order $p \geq k$ has $pk - \frac{k(k+1)}{2}$ edges.

The minimal dominating graph $MD(G)$ of G is a intersection graph on the minimal dominating sets of vertices of G. This concept was introduced by Kulli and Janakiram [2].

In [5], the concept of common minimal dominating graph $CD(G)$ of G was defined as the graph having the same vertex set as G with two vertices adjacent if there is a minimal dominating set containing them.

The concept of vertex minimal dominating graph $M_v D(G)$ of G was introduced in [3] as the graph having $V(M_v D(G)) = V(G) \cup S(G)$, where $S(G)$ is the set of all minimal dominating set of G with two vertices u, v adjacent if they are adjacent in G or $v = S_1$ is a minimal dominating set containing u.

In [4], the concept of dominating graph $D(G)$ of G as the graph with $V(D(G)) = V(G) \cup S(G)$, where $S(G)$ is the set of all minimal dominating sets of G with two vertices $u, v \in V(D(G))$ adjacent, if $u \in V$ and $v = S_1$ is a minimal dominating set containing u.

In this paper, we introduce the concept of edge minimal dominating graph $Ed(G)$ of a graph G, with $V(Ed(G)) = E \cup D$, where $E = E(G)$ is edge set of G and D is the set of all minimal dominating sets of G with two vertices $u, v \in V(Ed(G))$ adjacent if either they are adjacent edges in G or $v = D_1$ is a minimal dominating set of G containing vertices incident with $u \in E$ in G.

In Fig.1, a graph G and its edge minimal dominating graph $Ed(G)$ are shown.

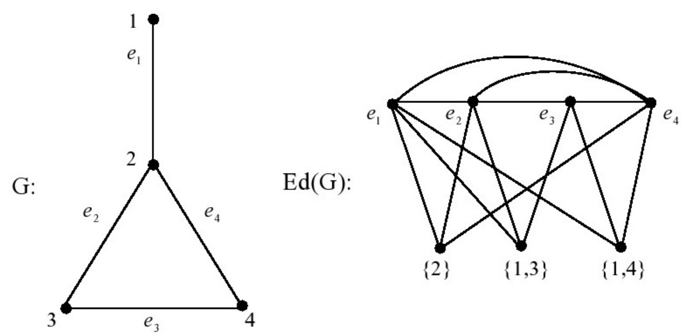

Figure 1.

The following results are useful to prove our next results.

Theorem A [1]. If G is a (p,q) graph whose vertices have degree d_i, then $L(G)$ has q vertices and q_L edges, where $q_L = -q + \frac{1}{2}\sum d_i^2$.

Theorem B [1]. A graph G is eulerian if and only if every vertex is of even degree.

Remark 1. For any graph G, $L(G)$ is an induced subgraph of $Ed(G)$.

Remark 2. For any graph G, $D = \{D_1, D_2, \ldots, D_n\}$ is independent set of $Ed(G)$.

Results

Lemma 1. If G is any (p,q) graph, then

$$\deg_{Ed(G)}(D_i) = \begin{cases} \text{sum of the degrees of each elements of minimal dominating} \\ \text{set } D_i \text{ which is independent in } G. \\ \\ \text{sum of the degrees of each elements of minimal dominating} \\ \text{set } D_i \text{ which is not independent in } G - \text{number of pairs of adjacent} \\ \text{vertices of } D_i \text{ in } G. \end{cases}$$

where D_i's, $i = 1, 2, \ldots, n$ are the minimal dominating sets of G..

Proof: We consider the following cases.

Case 1. Let D_i, for some i, be any minimal dominating set of G, and if it is independent set of G, then by definition of $Ed(G)$, D_i is adjacent with element $u \in E$ in $Ed(G)$, if D_i contains the vertices incident with the edge u in G. Hence degree of D_i in $Ed(G)$ is equal to the sum of the degree's of elements of D_i in G. Therefore,

$$\deg_{Ed(G)}(D_i) = \text{sum of the degrees of each elements of } D_i \text{ which is independent in } G.$$

Case 2. Let D_i, for some i, be the any minimal dominating set of G, and it is not independent. Suppose there exist m pair of elements in D_i, which are adjacent in G. Then the degree of D_i in $Ed(G)$ is the difference of the sum of the degree's of each elements of D_i in G and the number of pair of elements of D_i, which are adjacent in G (*i.e.*, m). □

Lemma 2. If G is any (p,q) graph, then

$$\deg_{Ed(G)}(e_i) = \text{the number of minimal dominating sets of } G \text{ containing the vertices incident with } e_i \text{ in } G + (\text{edge degree of } e_i \text{ in } G) - 2.$$

where e_i's, $1 \le i \le q$, are the edges of G.

Proof: Clearly $L(G)$ is an induced subgraph of $Ed(G)$. By definition of $Ed(G)$, element $u \in E$ is adjacent with element $v \in D$, if the element v which corresponds to minimal dominating set of G, contains the vertices incident with edge, u in G. Hence the degree of the elements of E in $Ed(G)$ is given by,

$$\deg_{Ed(G)}(e_i) = \text{the number of minimal dominating sets of } G \text{ containing the vertices incident with } e_i \text{ in } G + (\text{edge degree of } e_i \text{ in } G) - 2.$$

Theorem 3. For any (p,q) graph G,

$$V(Ed(G)) = q + |D|$$

and

$$E(Ed(G)) = \frac{1}{2}\sum_{i=1}^{P} d_i^2 - q + \sum_{j=1}^{n} \deg(D_j)$$

where d_i is the number of edges incident with a vertex v_i in G and D_j's, $j = 1, 2, \ldots, n$ are the minimal dominating sets of G.

Proof: By definition of $Ed(G)$ of G, the number of vertices in $Ed(G)$ is given by,

$$V(Ed(G)) = \text{the number of edges of } G + \text{the number of minimal dominating sets of } G.$$

$$V(Ed(G)) = q + |D|.$$

Clearly $L(G)$ is an induced sub graph of $Ed(G)$. The number of edges in $Ed(G)$ is the sum of the edges of $L(G)$ and edges between the elements of D and E, which is equal to the sum of the degree's of D_j, $1 \le j \le n$. Therefore

$$E(Ed(G)) = \frac{1}{2}\sum_{i=1}^{P} d_i^2 - q + \sum_{j=1}^{n} \deg(D_j). \ \square$$

Theorem 4. For any graph G, $Ed(G)$ is connected.

Proof: We consider the following cases.

Case 1. Let G be a non-trivial connected graph, and $D_1, D_2, \ldots, D_n$ be the minimal dominating sets of G. By the definition of $Ed(G)$, $L(G) \subset Ed(G)$, which implies that $L(G)$ is connected in $Ed(G)$. And also each D_i in $Ed(G)$ is adjacent with at least one element of E. Hence $Ed(G)$ is connected.

Case 2. If G is disconnected, then we consider the following subcases.

Subcase 2.1. Suppose G is totally disconnected. Then G contains only one minimal dominating set, which implies $Ed(G) = K_1$. Therefore $Ed(G)$ is connected.

Subcase 2.2. Suppose G has at least two component each of size is greater than or equal to one. Then clearly $L(G)$ is disconnected. Since $D_1, D_2, \ldots, D_n$ are the minimal dominating sets of G. In $Ed(G)$, the components of $L(G)$ are connected through the minimal dominating sets. Similar argument follows for more than two components also. Hence $Ed(G)$ is connected.$\square$

Theorem 5. For any graph G of size ≥ 2,

$$g(Ed(G)) = \begin{cases} 3 & \text{if } P_3 \subseteq G \\ 4 & \text{otherwise.} \end{cases}$$

Proof: Let G be any graph. We consider the following cases.

Case 1. Suppose P_3 is a subgraph of G. Then by definition of $Ed(G)$, the edges e_1, e_2 are adjacent in G and there exist at least one minimal dominating set d in G, which contains the vertex v incident with both e_1 and e_2 in G. Thus, in *Ed(G)* the vertices e_1, d, e_2, e_1 form a cycle of length three. Therefore $g(Ed(G)) = 3$, if $P_3 \subseteq G$.

Case 2. Suppose G do not contains P_3 as a subgraph. Then we consider the following subcases.

Subcase 2.1. Suppose G is totally disconnected. Then $Ed(G) = K_1$.

Subcase 2.2. Suppose $G = nK_2$ with $n = 1$. Then $Ed(G) = P_3$. If $n > 1$, then by Theorem 6, $Ed(G)$ is complete bipartite. Since $Ed(G)$ is complete bipartite, it has no odd cycles. Also it is well known that for the complete bipartite (with $n > 1$) graphs, shortest cycle is C_4. Therefore, $g(Ed(G)) = 4$.

Subcase 2.3. Suppose $G = nK_2 \cup mK_1$ with $n > 1$. Then similar argument follows as in above subcase. □

Theorem 6. For any graph G, $Ed(G)$ is complete bipartite if and only if $G = nK_2$ or $nK_2 \cup mK_1$, for any integer m, n.

Proof : Suppose $G = nK_2$. Then no two edges in G are adjacent. Consequently in $Ed(G)$ the set E is independent, whose elements corresponds to the edges of G. And also by Remark 2, the set D is independent, whose elements corresponds to the minimal dominating sets of G. Since $G = nK_2$, each minimal dominating set of G contains one vertex from each edge of G. Hence by definition of $Ed(G)$, it follows that, every element of the set D is adjacent to all the elements of the set E of $Ed(G)$. Therefore the resulting graph $Ed(G)$ is complete bipartite.

Suppose $G = nK_2 \cup mK_1$. Then there are no more extra edges compare to $G = nK_2$. Therefore the adjacency between the vertices of $Ed(G)$ remains same as that of $G = nK_2$. But the only change is in cardinality of minimal dominating sets, that is we have same minimal dominating sets as in $G = nK_2$, with additional vertices (as many isolated vertices). Hence $Ed(G)$ is complete bipartite.

Conversely, suppose $Ed(G)$ is complete bipartite. Then $V(Ed(G))$ can be partitioned into subsets $V_1 = E$ and $V_2 = D$. Clearly V_1 and V_2 are independent.

Suppose, assume that $G \neq nK_2$ or $nK_2 \cup mK_1$. Then G may be connected. By definition of $Ed(G)$, $L(G) \subset Ed(G)$. This implies that $L(G)$ is connected in $Ed(G)$, a contradiction to our assumption that the set $E = V_1$ of $Ed(G)$ is independent.

Suppose G is totally disconnected. Consequently $Ed(G) = K_1$ again a contradiction. Hence $G = nK_2$ or $nK_2 \cup mK_1$. □

Theorem 7. For any graph G of size ≥ 1,

$$\chi(Ed(G)) = \begin{cases} \chi(L(G))+1 & \text{if the edges of } G \text{ are colored with } \chi(L(G)) \text{ colors, which are incident with elements of any minimal dominating set of } G. \\ \chi(L(G)) & \text{otherwise.} \end{cases}$$

Proof: Let G be any graph with size ≥ 1 and $\chi(L(G)) = K$. Clearly, $L(G) \subset Ed(G)$ and the set D in $Ed(G)$ is independent, whose elements corresponds to minimal dominating sets of G. To color the graph $Ed(G)$ we have to color vertices of $L(G)$ and elements of D. Since D is independent set of $Ed(G)$, while coloring $Ed(G)$, either we make use of the colors, which are used to color $L(G)$ or we should use one more new color. In particular, if the edges of G are colored with K colors, which are incident with elements of any minimal dominating set of G. Then in $Ed(G)$ we required one more new color, in addition to K colors to color the vertex, which corresponds to that minimal dominating set of G. Therefore in this case we required $K+1$ colors to color $Ed(G)$. □

Theorem 8. For any graph G, $Ed(G)$ is eulerian if and only if the following conditions are satisfied:

(i) If the degree of the edge e in G is even, then the number of minimal dominating sets, contains the vertices incident with e, should be even.

OR

If the degree of the edge e in G is odd, then the number of minimal dominating sets, contains the vertices incident with e, should be odd;

(ii) If the minimal dominating set of G is independent, then the sum of the degrees of the elements of that minimal dominating set of G, should be even.

OR

If the minimal dominating set of G is not independent, then the difference of the sum of the degrees, of the elements of that minimal dominating set of G, and the number of pairs of adjacent vertices of that minimal dominating set in G, should be even.

Proof : Suppose $Ed(G)$ is eulerian. On the contrary, if one of the given condition say (i) is not satisfied, then there exist an edge e of even degree in G and the number of minimal dominating sets containing the vertices incident with e is odd. Hence $Ed(G)$, has a vertex of odd degree, a contradiction.

Suppose there exist an edge e of odd degree in G, and the number of minimal dominating sets contains the vertices incident with e is even, then $Ed(G)$ has a vertex of odd degree, a contradiction. Therefore condition (i) holds.

If the given condition say (ii) is not satisfied. Then there exist, independent minimal dominating set d of G and the sum of the degrees of the elements of that minimal dominating set, d of G is odd. Therefore $Ed(G)$, has a vertex of odd degree, a contradiction.

Suppose there exist non independent minimal dominating set d of G, and the difference of the sum of the degrees of the elements of d, and number of pairs of adjacent vertices of d in G is odd. Then $Ed(G)$ has a vertex of odd degree, a contradiction. Therefore condition (ii) holds.

Conversely, suppose the given conditions are satisfied. Then, every vertex of $Ed(G)$ has even degree and hence $Ed(G)$, is eulerian. □

Theorem 9. For any graph G,

$$\kappa(Ed(G)) = \min\left\{\min_{1\le i\le n}\{\deg_{Ed(G)}(D_i)\},\ \min_{1\le j\le q}\{\deg_{Ed(G)}(e_j)\}\right\}$$

Proof: We consider the following cases.

Case 1. Let $x = D_i$ for some i, be the minimal dominating set of G and it has the minimum degree among all the vertices of $Ed(G)$, which corresponds to the minimal dominating sets of G. If degree of x is less than all other vertices of $Ed(G)$, then by deleting the vertices of $Ed(G)$, which are adjacent with x, results in a disconnected graph. Thus,

$$\kappa(Ed(G)) = \min_{1\le i\le n}\{\deg_{Ed(G)}(D_i)\}.$$

Case 2. Let $y = e_j$ for some j, be the edge of G and it has the minimum degree among all the vertices of $Ed(G)$, which corresponds to the edges of G. If degree of y is less than all other vertices of $Ed(G)$, then by deleting the vertices of $Ed(G)$, which are adjacent with y, results in a disconnected graph. Thus,

$$\kappa(Ed(G)) = \min_{1\le j\le n}\{\deg_{Ed(G)}(e_j)\}.$$

By combining above two cases we get,

$$\kappa(Ed(G)) = \min\left\{\min_{1\le i\le n}\{\deg_{Ed(G)}(D_i)\},\ \min_{1\le j\le q}\{\deg_{Ed(G)}(e_j)\}\right\}. \ \square$$

Theorem 10. For any graph G,

$$\lambda(Ed(G)) = \min\left\{\min_{1\le i\le n}\{\deg_{Ed(G)}(D_i)\},\ \min_{1\le j\le q}\{\deg_{Ed(G)}(e_j)\}\right\}.$$

Proof: We consider the following cases.

Case 1. Let $x = D_i$ for some i, be the minimal dominating set of G, and it has the minimum degree among all the vertices of $Ed(G)$, which corresponds to the minimal dominating sets of G. If degree of x is less than all other vertices of $Ed(G)$, then by deleting the edges in $Ed(G)$, which are incident with x. The resulting graph will be disconnected. Thus,

$$\lambda(Ed(G)) = \min_{1\le i\le n}\{\deg_{Ed(G)}(D_i)\}.$$

Case 2. Let $y = e_j$ for some j, be the edge of G and it has the minimum degree among all the vertices of $Ed(G)$, which corresponds to the edges of G. If degree of y is less than all other vertices of $Ed(G)$, then by deleting the edges in $Ed(G)$, which are incident with y. The resulting graph will be disconnected. Thus,

$$\lambda(Ed(G)) = \min_{1\le j\le q}\{\deg_{Ed(G)}(e_j)\}.$$

By combining above two cases we get,

$$\lambda(Ed(G)) = \min\left\{\min_{1\le i\le n}\{\deg_{Ed(G)}(D_i)\},\ \min_{1\le j\le q}\{\deg_{Ed(G)}(e_j)\}\right\}. \quad \square$$

Theorem 11. For any graph G, $\alpha_0(Ed(G)) = q$, where q denotes the number of edges of G.

Proof : Let E be the set of vertices of $Ed(G)$, which corresponds to the edges of G. By definition of $Ed(G)$, elements of E covers all the edges of $Ed(G)$. Hence the set E is a vertex cover of $Ed(G)$. Now we have to show that E is the minimal vertex cover of $Ed(G)$. Suppose we drop any one vertex from the set E. Then clearly, E is not vertex cover of $Ed(G)$. Therefore, the set E is the minimal vertex cover of $Ed(G)$.

Hence, $\alpha_0(Ed(G)) = |E| = q.$ □

Theorem 12. For any graph G, $\beta_0(Ed(G)) = |D|$.

Proof : Let D denotes the set of vertices of $Ed(G)$, which corresponds to the minimal dominating sets of G. Clearly, the set D of $Ed(G)$ is independent. Now we have to show that the set D is maximal independent set of $Ed(G)$.

Suppose there exist maximal independent set D' contains D. Then D' contains some other vertices of $Ed(G)$ with addition to the vertices of D. By definition of $Ed(G)$, there exist at least one pair of adjacent vertices in D', a contradiction. Hence the set D of $Ed(G)$ is the maximal independent set.

Therefore, $\beta_0(Ed(G)) = |D|.$ □

We now determine line covering number α_1 of the edge minimal dominating graph.

Theorem 13. For any graph G, $|D| \le \alpha_1(Ed(G)) \le |E|$.

Proof : Let E denotes the set of vertices of $Ed(G)$, which corresponds to the edges of G and let D denotes the set of vertices of $Ed(G)$, which corresponds to the minimal dominating sets of G.

We consider the following cases depending upon the cardinality of the sets E and D of $Ed(G)$.

Case 1. Suppose $|D| \geq |E|$. Then clearly, D is independent set of $Ed(G)$. Therefore to cover the elements of D, we have to select $|D|$ number of edges, which also covers the other vertices of $Ed(G)$. Let us denote this set of edges by $N = \{e_i : 1 \leq i \leq q\}$. Clearly this set N form the edge cover for $Ed(G)$. Now we have to show that the set N is the minimal edge cover of $Ed(G)$.

Suppose we drop any one edge from the set N. Then clearly, the set N is not line cover of $Ed(G)$. Therefore set N is the minimal line cover of $Ed(G)$. Hence, $\alpha_1(Ed(G)) = |D|$.

Case 2. Suppose $|D| < |E|$. Then $|D|$, number of edges are not enough to cover all the vertices of $Ed(G)$. Therefore we required some more edges, with addition to $|D|$ number of edge. Hence, $|D| < \alpha_1(Ed(G))$.

Also it is easy to see that, $|E|$ number of edges of $Ed(G)$ covers all the vertices of $Ed(G)$. Therefore, $\alpha_1(Ed(G)) \leq |E|$. □

Theorem 14. $Ed(G)$ is *k*-tree of order $k+2$, $(k \geq 1)$, if and only if $G = K_{1,K}$ or $K_{1,K} \cup mK_1$, where m is any integer.

Proof: Suppose $G = K_{1,K}$. Then clearly, G has exactly two disjoint minimal dominating sets, and elements of these two minimal dominating sets are incident with all the edges of G. By definition of $Ed(G)$, it follows that the vertices which corresponds to the minimal dominating sets of G in $Ed(G)$, are adjacent to all the vertices of $L(G) = K_K$, which clearly gives the *k*-tree of order $k+2$.

Suppose $G = K_{1,K} \cup mK_1$. Then, there are no more extra edges, compare to $G = K_{1,K}$. Therefore, the adjacency between the vertices of $Ed(G)$ remains same as that of $G = K_{1,K}$. But, the only change is in cardinality of minimal dominating sets of G, that is we have same minimal dominating sets as in $G = K_{1,K}$, with additional vertices (m isolated vertices). Hence $Ed(G)$ is *k*-tree of order $k+2$.

Conversely, suppose $Ed(G)$ is *k*-tree of order $k+2$. Then the vertices of degree *k* in $Ed(G)$, corresponds to the minimal dominating set of G and we have only two vertices of degree *k* in $Ed(G)$. Clearly these vertices form a set D in $Ed(G)$, and remaining all other vertices of $Ed(G)$, which corresponds to the edges of G form the set E of $Ed(G)$. Hence,

$$Ed(G) - D = L(G)$$

$$\Rightarrow \quad Ed(G) - D = K_K$$

$$\Rightarrow \quad L(G) = K_K$$

$$\Rightarrow \quad G = K_{1,K} \text{ or } K_{1,K} \cup mK_1$$ □

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