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DETERMINISTIC FRACTALS

VISHWA JEET BANSAL

Assistant Professor, Department of Mathematics, Govt. College Paonta Sahib (H.P)

AND

DR. A.K. VYAS

Faculty, Department of Mathematics, Jodhpur National University, Rajasthan

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In this paper it is shown that a collection comp (*X*), of non – empty compact subsets of a metric space *X*, itself becomes a metric space with respect to a metric suitably defined on it. If *X* is taken to be complete then interestingly this space turns out to be a complete metric space. While proving the space to be complete, there comes a very strong and useful result which enables to extend a Cauchy subsequence of points of *X* to a Cauchy sequence in *X*. Definition to a contraction mapping on the space comp (*X*) is also given. And since Banach's Contraction Mapping Principle ensures that a contraction map has a unique fixed point, a Deterministic Fractal is obtained at the end.

KEY WORDS : Distance between two sets, Comp (*X*), *IFS*, Attractor, Fixed Point, Deterministic Fractal etc.

INTRODUCTION

We start the proceedings by defining two real valued mappings α and θ_B on a subset of a metric space *X*. It is proved that these mappings are distance decreasing and uniformly continuous. It is established that for a compact set, its distance from a point is gettable. The distance between any two sets is defined and it is seen that the distance of a set from the second is not equal to the distance of second from the first set, in general. Thereafter we prove several important propositions concerning maximum distance between two non-empty compact subsets of *X*. A map d^L is then defined from comp (*X*) x comp (*X*) into *IR*. We prove that (comp (*X*), d^L) is a metric space. We successfully reduce the distance between two sets as per our requisition. The space is proved to be complete and finally after defining IFS an attractor and a deterministic fractal is obtained.

PRELIMINARIES

Definition 1.1. Let *X* be a metric space with a metric *d*. Let $f : X \to X$ be a map. Then *f* is called a contraction mapping if there exists a real s, $0 \leq s \leq 1$, such that $d(f(x), f(y)) \leq s\, d(x, y)\ \forall\ x, y \in X$.

Any such number $\boldsymbol{s}$ is called a contractivity factor for f. If $\boldsymbol{s} = 0$, then f is a constant map and will be called a trivial contraction mapping and if $\boldsymbol{s} > 0$, then f is called a non-trivial contraction mapping.

Definition 1.2. Let X and Y be two metric spaces. A function $f : X \to Y$ is called a distance decreasing map if $d(f(x_1), f(x_2)) \leq \boldsymbol{s}\, d(x_1, x_2)\ \forall\ x_1, x_2 \in X$.

Notation 1.3. J_N will denote the set $\{1, 2, 3, 4, \ldots, n\}$, $n \in IN$. If $x, y \in IR$, then the maximum of x and y will be represented as $x \vee y$.

Remark 1.4. Let X be a metric space. Let $f : X \to X$ be a non- trivial contraction mapping. Then f is called a distance decreasing map.

Definition 1.5. Let X be a metric space. Let $\emptyset \neq B \subset X$ be compact. We define $d(x, B) = \inf \{d(x, y) : y \in B\}$ and is called as the distance of x from B.

Remark 1.6. (i) $d(x, B) \leq d(x, y)\ \forall\ y \in B$.

(ii) Let X be a metric space. Let $\emptyset \neq B_1 \subset B_2 \subset$ X be compact. If $x \in X$, then $d(x, B_2) \leq d(x, B_1)$.

Proposition 1.7. Let α, β be any two subsets of IR. Let for each $a \in \alpha$, $\exists$ some $b \in \beta$ such that $a \leq b$, then $\sup \alpha \leq \sup \beta$.

Proof. Let $a \in \alpha$, then $\exists$ some $b \in \beta$ such that $a \leq b$. But $b \leq \sup \beta\ \forall\ a \in \alpha$. This implies $\sup \alpha \leq \sup \beta$.

Corollary 1.8. If $x \leq x_0$ and $y \leq y_0$ for x, x_0, y and $y_0 \in IR$, then $x \vee y \leq x_0 \vee y_0$.

Proof. Let $\alpha = \{x, y\}$ and $\beta = \{x_0, y_0\}$. Then, we have, $\sup \alpha \leq \sup \beta \Rightarrow x \vee y \leq x_0 \vee y_0$.

Remark 1.9. Let $A, B \subset IR$ be two non-empty sets that are bounded above. Then sup ($A \cup$ B) is bounded above and

(i) If $A \subset B$ then $\sup A \leq \sup B$

(ii) $\sup (A \cup B) \leq \sup A + \sup B$

Proposition 1.10. A sequence of integers is Cauchy iff it is almost constant.

Remark 1.11. Let $n \in IN$. Then $\sum_{i=1}^{\infty} N/(N+1)^i$ is convergent.

Proof. $N + 1 \geq 2 \Rightarrow 1/N + 1 \geq 1/2$. This implies $1/(N + 1)^I \leq 1/2^i$ for each $I \in IN$. Now by comparison test $\Sigma\ 1/(N + 1)^I$ is convergent as $\Sigma\ 1/2^i$ is convergent. Hence $\sum_{i=1}^{\infty} N/(N+1)^i$ is convergent.

Remark 1.12. Let $K, N \in IN$. Then $\sum_{i=K+1}^{\infty} N/(N+1)^i = 1/N\ .\ (N + 1)^K$.

Proof. $\sum_{i=K+1}^{\infty} N/(N+1)^i = [1/(N+1)^{K+1}]./[1 - 1/(N+1)] = (N+1)/N.\ (N+1)^{K+1} = 1/N.\ (N+1)^K$.

Remark 1.13. Let $\sum_n a_n$ be convergent, where $a_n \geq 0\ \forall\ n \in IN$. If $\sum_n a_n = 0$ then $a_n = 0$ $\forall\ n \in IN$.

Remark 1.14. If $\Sigma\,|\,a_n\,|$ is convergent, then for each $j \in IN$, $|\,a_j\,| \leq \Sigma\,|a_n|$.

CODE SPACE

Definition 2.1. Let $N \in IN$. Let $K_N = \{0, 1, 2, 3, \ldots, N - 1\}$. We, now, define a word on K_N as : Let $x = x_1 x_2 x_3 \ldots$ where $x_\alpha \in K_N$ for all $\alpha \in IN$. Sometimes this word is called a semi-infinite word. Let X be the set of all such words, that is $X = \{x : x = x_1\, x_2\, x_3\, \ldots, x_\alpha \in K_N$, for all $\alpha \in IN\}$. Then, $x = y$ iff $x_\alpha = y_\alpha \forall\ \alpha \in IN$.

Remark 2.2. Let x and y be any two semi-infinite words in X as defined above, then $\sum |\, x_i - y_i\,|/(N+1)^I\ \forall\ x, y \in X$, is convergent.

Proof. Clearly, $\sum |\, x_i - y_i\,|/(N+1)^I\ \forall\ x, y \in X$, is convergent. Since $|\, x_i - y_i\,| \le N$, $|\, x_i - y_i\,|/(N+1)^I \le N/(N+1)^i$ and therefore by comparison test, $\sum |\, x_i - y_i\,|/(N+1)^I\ \forall\ x, y \in X$, is convergent.

Proposition 2.3. Let $x = x_1\, x_2\, x_3$ ……. and $y = y_1\, y_2\, y_3\, \ldots$, be two words in X. Let a function $d : X\, xX \to IR$ be defined as $d\,(x, y) = \sum |\, x_i - y_i\,|/(N+1)^I\ \forall\ x, y \in X$. Then d is a metric on X.

Definition 2.4. We define this metric space, over the set K_N, as defined in 2.3, as code space and written as C_d.

Lemma 2.5. Let $x, y \in C_d$ and $K \subseteq IN$. Then, $\sum_{i=K+1}^{\infty} N/(N+1)^i \le 1/(N+1)^K$.

Lemma 2.6. Let $x = x_1\, x_2\, x_3\, \ldots$ and $y = y_1\, y_2\, y_3$ …. be any two words in C_d and let $y^m = y_1^m\, y_2^m\, y_3^m\, \ldots$ where $m \in IN$. Suppose that there exists some $k \in IN$ such that $\forall\ n \ge k$, $y_i^n = x_i \forall\ i,\ 1 \le i \le n$ then $y_n \to$ x.

Proof. Let $\in\ > 0$ be given, then there exists some $n_0 \in IN$, $n_0 \ge k$ such that

$$1/(N+1)^{n_0} < \in \qquad \ldots (1)$$

Let $m > n_0$, we have $y^m = y_1^m\, y_2^m\, y_3^m\, \ldots\, y_{n0}^m\, y_{n0+1}^m$ ….. y_m^m …. And $y_i^m = x_i \forall\ i,\ 1 \le i \le n_0$. Now, $d\,(y^m, x) = \sum |\, y_i^m - x_i\,|/(N+1)^I \le 1/(N+1)^{n_0}$ by using 2.5 and therefore by (1) we have $d\,(y^m, x) < \in\ \forall\ m > n_0$, and hence $y_n \to x$. □

Corollary 2.7. Let $x \in C_d$ and $\in\ > 0$, then $\exists$ some $y \in C_d$, $y \ne x$, such that $d\,(x, y) < \in$.

Proof. Let $x \in C_d$ and let $\in\ > 0$ be given. There exists $n_0 \in IN$ such that $1/(N+1)^{n_0} < \in$. Let $y \in C_d$ be such that $y_i = x_i \forall\ i,\ 1 \le i \le n_0$ but, $y_{n_0+1} \ne x_{n_0+1}$, then by definition, $y \ne x$. Now, by 2.5, $\sum_{i=n_0+1}^{\infty} N/(N+1)^i \le 1/(N+1)^{n_0} < \in$ and therefore, we have, $d\,(x, y) < \in$. Hence, the result stands proved. □

Corollary 2.8. Let $x \in C_d$ and $a \in K_N$, and also let $y^n = x = x_1\, x_2\, x_3\, \ldots\, x_n\, aa\, a\, \ldots$ then $y^n \to x$.

Proof. Let $n \in IN$ then $y_i^m = x_i\ \forall\ i,\ 1 \le I \le n$, then by 2.6, $y^n \to x$. □

Corollary 2.9. Let $x \in C_d$, and let $a \in K_N$. Also, let $y^n = a\ aa\ \ldots\ \ldots\ x_1\, x_2\, x_3\, \ldots\, x_n$.., then $y^n \to z = a\ aa$…..

Proof. Let us consider a word $z\ (= z_1\, z_2\, z_3\, \ldots)$ in C_d where $z_i = a\ \forall\ i \in IN$. Now, $y_i^n = z_i = a$ $\forall\ i,\ 1 \le I \le n$ and thence by 2.6, $y^n \to z = a\ aa$….. □

Lemma 2.10. Let $\{x^n\}$ be a sequence of points of C_d such that each of its sequence of points is almost constant, then the sequence $\{x^n\}$ is convergent.

Lemma 2.11. Let $\{x^n\}$ be a Cauchy sequence of points of the space C_d. Then, for $j \in IN$, $\{x_j^n\}$ is a Cauchy sequence of integers.

Proof. Let $j \in IN$, and let $\in\ > 0$ be given. Since $\{x^n\}$ is a Cauchy sequence of points of the space C_d, $\exists\ n_0 \in IN$ such that $d\,(x^n, x^m) < \in/(N+1)^j \le \sum |\, x_i^n - x_i^m\,|/(N+1)^I \le \in/(N+1)^j$ $\Rightarrow |\, x_j^n - x_j^m\,| < \in \forall\ n, m \ge n_0$. Hence, $\{x_j^n\}$ is a Cauchy sequence of integers. □

Proposition 2.12. A sequence in a given code space converges iff each of its sequence of co-ordinates is almost constant.

Proof. Let $\{x^n\}$ be a sequence of points of C_d. Then, for $n \in IN$, we have, $x^n = x_1{}^n x_2{}^n x_3{}^n \ldots$ Suppose for each i, $I \in IN$, $\{x_i{}^n\}$ is almost constant. Then by 2.10, $\{x^n\}$ is convergent. Conversely, let $\{x^n\}$ be convergent in C_d. Then $\{x^n\}$ is a Cauchy sequencein C_d. If $j \in IN$ then $\{x_j{}^n\}$ is a Cauchy sequence of j^{th} co-ordinates, and therefore by 2.11, $\{x_j{}^n\}$ is a Cauchy sequence of integers. Hence by 1.10, the result follows. □

Theorem 2.13. C_d is complete.

Proof. Let $\{x^n\}$ be a sequence of points of C_d. If $j \in IN$ then by 2.11, $\{x_j{}^n\}$ is a Cauchy sequence of integers. Again, by 1.10 $\{x_j{}^n\}$ is almost constant, for each $j \in IN$ kept fixed. Finally, using 2.12, we see that $\{x^n\}$ is convergent in C_d. Hence, the code space C_d is complete. □

Proposition 2.14. Each point of the code space, C_d, is a limit point of C_d. In other words $C_d \subseteq \text{der}\,(C_d)$.

Proof. Let $x \in C_d$, and let $\in > 0$ be given. Then by 2.7, there exists some $y \in C_d$ such that $y \neq x$ and $d\,(y, x) < \in$. This implies x is a limit point of $C_d \Rightarrow x \in \text{der}\,(C_d) \Rightarrow C_d \subseteq \text{der}\,(C_d)$. □

Theorem 2.15. The code space, C_d, is perfect.

Proposition 2.16. Let $a \in K_N$. Let z (= aaa . . .) be a word in C_d. Define a map $g : C_d \to C_d$ as $g\,(x) = a\, x_1\, x_2\, x_3 \ldots$ where $x = x_1\, x_2\, x_3 \ldots \in C_d$. Then

(i) g is a non-trivial contraction map with contractivity factor $1/(N + 1)$

(ii) z (= $a\, aa$ . . .) is a fixed point of g.

In order to begin our main work, we define the following two maps

Definition. Let X be a metric space and let $x_0 \in X$. For $A \subseteq X$, we define, $\alpha : A \to IR$ as $\alpha\,(x) = d\,(x_0, x)\ \forall\ x \in A$. Also, for $\varnothing \neq B \subseteq X$, we define, $\theta_B : A \to IR$ as $\theta_B\,(x) = d\,(x, B)\ \forall\ x \in A$.

Proposition 3.1. α is a distance decreasing (↓) map.

Proof. If $x, x' \in A$, then $\alpha\,(x) = d\,(x_0, x) \leq d\,(x_0, x') + d\,(x', x) \Rightarrow d\,(x_0, x) - d\,(x_0, x') \leq d\,(x', x) = d\,(x, x')$, and interchanging the role of x and x', we get $d\,(x_0, x) \leq d\,(x, x')$, which gives $-(d\,(x_0, x) - d\,(x_0, x')) \leq d\,(x, x')$. Combining these two we get, $|\,(d\,(x_0, x) - d\,(x_0, x')\,| \leq d\,(x, x')$ and hence, α is a distance ↓ map.□

Corollary 3.2. α is uniformly continuous.

Proof. Since every distance ↓ map is uniformly continuous, we have, α is uniformly continuous.

Corollary 3.3. α is continuous.

Proof. Since every uniformly continuous map is continuous, therefore, α is continuous.

Proposition 3.4. θ_B is a distance decreasing (↓) map.

Proof. Let $x, x' \in A$ and $y \in B$. Then by definition of θ_B we have, $d\,(x, B) \leq d\,(x, y) \leq d\,(x, x') + d\,(x', y) \Rightarrow d\,(x, B) \leq d\,(x, x') + \inf\,\{d\,(x', y) : y \in B\} \Rightarrow d\,(x, B) \leq d\,(x, x') + d\,(x', B) \Rightarrow d\,(x, B) - d\,(x', B) \leq d\,(x, x')$ …(1)

Interchanging the roles of x and x', we get $d\,(x', B) - d\,(x, B) \leq d\,(x', x)$ …(2)

By (1) and (2), we get $| d(x, B) - d(x', B) | \leq d(x, x') \Rightarrow | \theta_B(x) - \theta_B(x') | \leq d(x, x')$. Hence, θ_B is a distance decreasing (↓) map. □

Corollary 3.5. θ_B is uniformly continuous.

Proof. Since every distance ↓ map is uniformly continuous, we have, θ_B is uniformly continuous.

Corollary 3.6. θ_B is continuous.

Proof. Since every uniformly continuous map is continuous, therefore, θ_B is continuous.

Proposition 3.7. Let $\varnothing \neq A \subseteq X$ be compact and let $\varnothing \neq B \subseteq X$. Then $\theta_B(A) = \{d(x, B) : x \in A\}$ is a bounded subset of IR.

Proof. Proof follows immediately as every compact subset of a metric space is closed and bounded subset of IR.

We now show that the distance of a point from a compact set is gettable

Proposition 3.8. Let $\varnothing \neq B \subseteq X$ be compact. Let $x_0 \in X$. Then there exists $x_0 \in B$ such that $d(x_0, B) = d(x_0, y_0)$.

Proof. By 3.3, $\alpha : B \rightarrow IR$ is continuous, and therefore inf $\alpha(B)$ exists and hence ∃ some $y_0 \in B$ such that $\alpha(y_0) = \inf \alpha(B) = \inf \{\alpha(y) : y \in B\} = \{d(x_0, y_0) : y \in B\} = d(x_0, B)$. Thus $d(x_0, B) = d(x_0, y_0)$. Hence the result stands proved. □

Now, we define the distance between two non – empty sets

Definition. Let $A, B \subseteq X$ be non-empty. Then $d(A, B)$, the distance of set A from set B is defined as $d(A, B) = \sup \{d(x, B) : x \in A\}$ in IR.

Remark 3.9. $d(A, B) \neq d(B, A)$, in general.

Proof. Let $A \subseteq B$ be such that cl $(A) \subset$ cl (B), cl $(A) \neq$ cl (B). Then clearly, $d(A, B) = 0$. Now, we take $B = \{0 \leq x \leq 1 : x \in IR\}$ and $A = \{0 \leq x \leq 1/2 : x \in IR\}$. Then $d(B, A) = \sup \{d(y, A) : y \in B\} \geq d(1, A) > 0$ as $1 \notin$ cl (A) and $1 \in B$. Hence, $d(A, B) \neq d(B, A)$, in general.

It is pertinent to note that the order of the two sets matters while calculating distance between them

Remark 3.10. We note that $d(A, B) = \sup \theta_B(A)$. In general, $d(A, B)$ is a non-negative extended real number. But, if A is compact then clearly, $\sup \theta_B(A)$ is a real number. Therefore, if A is compact then $d(A, B)$ is a real number.

We require the following proposition which will be used as a strong tool in establishing some useful results

Proposition 3.11. Let $A, B \subseteq X$ be non-empty. Then,

(i) $d(x_0, B) \leq d(A, B) \ \forall \ x_0 \in A$.

(ii) $A \subseteq B \Rightarrow d(A, B) = 0$.

(iii) $d(A, B) = 0 \Rightarrow A \subseteq$ cl (B).

(iv) $d(A, B) = 0 = d(B, A) \Rightarrow A = B$ if A and B are compact.

Proof. (i) If $x_0 \in A$ then $d(x_0, B) \leq \sup \{d(x_0, B) : x_0 \in A\} = d(A, B)$, and so $d(x_0, B) \leq d(A, B) \ \forall \ x_0 \in A$.

(ii) Let $x \in A$ then $x \in B$ and therefore $d(x, B) = 0$. This implies $\sup \{d(x, B) : x \in A\} = 0 \Rightarrow d(A, B) = 0$.

(iii) Let $d(A, B) = 0$ then $\sup \{d(x, B) : x \in A\} = 0$. Let $x \in A$ then $d(x, B) = 0$ and so $x \in \mathrm{cl}(B)$. Hence, $A \subseteq \mathrm{cl}(B)$.

(iv) By (iii), $d(A, B) = 0 \Rightarrow A \subseteq \mathrm{cl}(B)$ where as $d(B, A) = 0 \Rightarrow B \subseteq \mathrm{cl}(A)$. This implies $\mathrm{cl}(A) = \mathrm{cl}(B)$. But A and B are compact and so $A = B$. □

For a given metric space X, by comp (X) we shall denote the set of all non-empty compact subsets of X, In fact, this is comp (X) only which shall be focused at length during the whole work

Proposition 3.12. Let $\varnothing \neq A_1 \subseteq A_2 \in \mathrm{comp}(X)$. Then $d(A_1, B) \leq d(A_2, B)\ \forall\ \varnothing \neq B \subseteq X$.

Proof. Let $\alpha = \{d(x, B) : x \in A_1\}$ and $\beta = \{d(x, B) : x \in A_2\}$. Since $A_1 \subseteq A_2$ and $\alpha \subseteq \beta$, therefore clearly $\sup \alpha \leq \sup \beta$. This implies that $d(A_1, B) \leq d(A_2, B)\ \forall\ \varnothing \neq B \subseteq X$.□

Proposition 3.13. Let $A \in \mathrm{comp}(X)$. Let $\varnothing \neq B_1 \subseteq B_2 \subseteq X$. Then $d(A, B_2) \leq d(A, B_1)$.

Proof. Let $\alpha = \{d(x, B_2) : x \in A\}$ and $\beta = \{d(x, B_1) : x \in A\}$.Then for any $d(x, B_2) \in \alpha$, clearly, $d(x, B_2) \leq d(x, B_1)$ and therefore $\sup \alpha \leq \sup \beta$. This implies that $d(A, B_2) \leq d(\mathrm{A}, B_1)$. □

Proposition 3.14. Let X be a metric space and $A, B, K \in \mathrm{comp}(X)$. Then

(i) $d(A \cup B, K) \geq d(A, K) \vee d(B, K)$

(ii) $d(A \cup B, K) \leq d(A, K) \vee d(B, K)$

(iii) $d(A \cup B, K) = d(A, K) \vee d(B, K)$

Proof. (i) Since $A \subseteq A \cup B$, therefore by 3.12 $d(A, K) \leq d(A \cup B, K)$ and similarly $d(B, K) \leq d(A \cup B, K)$. Thus, $d(A, K) \vee d(B, K) \leq d(A \cup B, K)$ or $d(A \cup B, K) \geq d(A, \mathrm{K}) \vee d(B, K)$

(ii) Let $x \in A \cup B$. If $x \in A$ then by 3.11 (i), $d(x, K) \leq d(A, K) \leq d(A, K) \vee d(B, K) \Rightarrow d(x, K) \leq d(A, K) \vee d(B, K)\ \forall\ x \in A \cup B$. This implies that $\sup \{d(x, K) : x \in A \cup B\} \leq d(A, K) \vee d(B, K)$ and hence $d(A \cup B, K) \leq d(A, K) \vee d(B, K)$

(iii) By (i) and (ii), $d(A \cup B, K) = d(A, K) \vee d(B, K)$. □

Proposition 3.15. If $A, B \in \mathrm{comp}(X)$, then $\exists\ x' \in A$ and $y' \in B$ such that

$$d(A, B) = d(x', y').$$

Proof. By 3.6, $\theta_B : A \to IR$ is continuous and also by 3.10, $d(A, B) = \theta_B(A) = \sup \{\theta_B(x) : x \in A\}$. This implies $d(A, B) = d(\mathrm{x}', B)$. Now, by 3.8, $\exists\ y' \in B$ such that $d(x', B) = d(x', y')$ and hence $d(A, B) = d(x', y')$. □

Lemma 3.16. Let X be a metric space and $A, B, K \in \mathrm{comp}(X)$. Then $d(A, B) \leq d(A, K) + d(K, B)$.

Proof. Let $a \in A$ and $\mathrm{y} \in K$ be fixed. For $b \in B$, clearly, $d(a, B) \leq d(a, b) \leq d(a, y) + d(y, \mathrm{b})$. This implies $d(a, B) \leq \inf \{d(a, y) + d(y, b) : b \in B\} = d(a, y) + \inf \{d(y, b) : b \in B\} \Rightarrow d(a, B) \leq d(a, y) + d(y, B) \leq d(a, y) + d(K, B)$, using 3.11 (i). Now, $d(a, B) \leq \inf \{d(a, y) + d(K, B) : y \in K\} = \inf \{d(a, y) : y \in K\} + d(K, B) = d(a, K) + d(K, B) \leq d(A, K) + d(K, B)$ using 3.11 (i).This implies that $\sup \{d(a, B) : a \in A\} \leq d(A, K) + d(K, B)$ and thus $d(A, B) \leq d(A, K) + d(K, B)$.□

We define a function d^L*: comp (X) x comp (X)* $\to$ *IR as given below*

Definition. Let $A, B \in$ comp (X), then $d^L(A, B) = d(A, B) \vee d(B, A)$. By 3.10, $d(A, B)$ and $d(B, A)$ are real numbers, and therefore d^L is a real number. □

Remark 3.17. Let $A, B \in$ comp (X), then $d(A, B) \leq d^L(A, B)$.

Theorem 3.18. d^L is a metric on comp (X).

Proof. Let $A, B \in$ comp (X), then $d^L(A, B)$ is a real number and $d(A, B) \geq 0$, $d(B, A) \geq 0$ $\Rightarrow d^L(A, \mathrm{B}) \geq 0$. Now, if $A = B$ then $d(A, A) = 0$ by 3.11 (ii) and therefore $d^L(A, A) = 0$ and if $d^L(A, B) = 0$ then $d(A, B) = 0 = d(B, A)$, and therefore by 3.11 (iv), $A = B$. Also, $d^L(A, B) = d(A, B) \vee d(B, A) = d(B, A) \vee d(A, B) = d^L(B, A)$. Next, let $A, B, K \in$ comp (X), then by 3.16, $d(A, B) \leq d(A, K) + d(K, B) \leq d^L(A, K) + d^L(K, B)$ and thus $d(A, B) \leq d^L(A, K) + d^L(K, B)$. Interchanging the role of A and B, we get $d(B, A) \leq d^L(B, K) + d^L(K, A) = d^L(A, K) + d^L(K, B)$, and hence we have, $d^L(A, B) \leq d^L(A, K) + d^L(\mathrm{K}, B)$. Thus, d^L is a metric on comp (X). □

Proposition 3.19. Let A, B, C and $D \in$ comp (X). Then

(i) $d(A \cup B, C \cup D) \leq d(A, C) \vee d(B, D)$

(ii) $d(A \cup B, C \cup D) \leq d(A, D) \vee d(B, C)$

(iii) $d(A \cup B, C \cup D) \leq d^L(A, C) \vee d^L(B, D)$

(iv) $d(C \cup D, A \cup B) \leq d^L(A, C) \vee d^L(B, D)$

(v) $d^L(A \cup B, C \cup D) \leq d^L(A, C) \vee d^L(B, \mathrm{D})$

Proof. Let A, B, C and $D \in$ comp (X). Then

(i) $d(A \cup B, C \cup D) \leq d(A, C \cup D) \vee d(B, C \cup D)$ by 3.14 (ii)

$\leq d(A, C) \vee d(B, D)$ by 3.13

(ii) $d(A \cup B, C \cup D) \leq d(A, C \cup D) \vee d(B, C \cup D)$

$\leq d(A, D) \vee d(B, C)$

(iii) $d(A, C) \leq d^L(A, C)$ and $d(B, D) \leq d^L(B, D)$ by 3.17, we have

$d(A, C) \vee d(B, D)\,) \leq d^L(A, C) \vee d^L(B, D)$ and thus $d(A \cup B, C \cup D)$

$\leq d^L(A, C) \vee d^L(B, D)$

(iv) Interchanging the role of A and C, B and D, by (iii), we get,

$d(C \cup D, A \cup B) \leq d^L(C, A) \vee d^L(D, B) = d^L(A, C) \vee d^L(B, D)$

and thus $d(C \cup D, A \cup B) \leq d^L(A, C) \vee d^L(B, D)$.

(v) By (iii) and (iv), we have

$d(A \cup B, C \cup D) \vee d(C \cup D, A \cup B) \leq d^L(A, C) \vee d^L(B, D)$

and hence, $d^L(A \cup B, C \cup D) \leq d^L(A, C) \vee d^L(B, D)$. □

Proposition 3.20. Let A, B and $C \in$ comp (X). Then $d^L(A, B) = d(x', y')$ for some $x' \in A$ and $y' \in B$.

Proof. We know that $d^L(A, B) = d(A, B) \vee d(B, A)$. Now, suppose $d^L(A, B) = d(A, B)$. By 3.15, $\exists$ some $x \in A$ and $y \in B$ such that $= d(A, B) = d(x, y)$. Thus we have $d^L(A, B) = d(x, y)$. Otherwise, $d^L(A, B) = d(B, A)$ and again by 3.15, $\exists$ some $x' \in A$ and $y' \in B$ such that $d(B, A) = d(y', x')$ and by symmetry $d(y', x') = d(x', y')$. Thus, $d(B, A) = d(x', y')$, where $x' \in A$ and $y' \in B$. Hence, $d^L(A, B) = d(x', y')$ for some $x' \in A$ and $y' \in B$. □

Notation. Let $A \subseteq X$ and $r \geq 0$ be a real number. We shall denote the set $\{y \in X : d(x, y) \leq r$ for some $x \in A\}$ by $A + r$.

Lemma 3.21. Let X be a metric space and let $M \subseteq X$ be compact. Then for $\in > 0$, $M + \in$ is closed.

Proof. We have $M + \in = \{y \in X : d(x, y) \leq \in$ for some $x \in M\}$. Let $\{y_n\}$ be a sequence of points of $M + \in$ such that

$$y_n \rightarrow y_0 \in X \qquad \text{... (1)}$$

Let $n \in IN$. Since $y_n \in M + \in$, $\exists$ some $x_n \in M$ such that

$$d(x_n, y_n) < \in \qquad \text{... (2)}$$

Now, $\{x_n\}$ is a sequence of points of M and M being compact is sequentially compact. This implies there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow x$, where x is a point of M. By (2),

$$D(x_{n_j}, y_n) < \in \qquad \text{... (3)}$$

This implies $d(x, y_0) < \in$ by using (1) and (3). This means $y_0 \in M + \in$ and hence $M + \in$ is closed. □

Remark 3.22. $A + r = \cup \{S[x, r] : x \in A\}$.

Proof. Let $x \in A$. Let $y \in S[x, r]$. Then, $d(x, y) \leq r$ and since $A + r = \cup \{S[x, r] : x \in A\}$, therefore $y \in A + r$. This implies $S[x, r] \subseteq A + r \ \forall \ x \in A \Rightarrow \cup \{S[x, r] : x \in A\} \subseteq A + r$. Now, let $y_r \in A + r$. Then $d(x, y_r) \leq r$ for some $x \in A \Rightarrow y_r \in S[x, r]$.

Hence, $\cup \{S[x, r] : x \in A\} = A + r$.

Lemma 3.23. Let $A, B \in$ comp (X) and $r \geq 0$ be a real number. Then $d(A, B) \leq r$ iff $A \subseteq \cup \{S[y, r] : y \in B\} = B + r$.

Proof. Suppose $d(A, B) \leq r$. Then sup $\{d(x, B) : x \in A\} \leq r$. Let $x \in A$, then $d(x, B) \leq r$. This implies that inf $\{d(x, y) : y \in B\} \leq r$. By 3.8, $y_0 \in B$ such that $d(x, B) = d(x, y_0) \Rightarrow d(x, y_0) \leq r$ which gives $x \in S[y_0, r]$. Thus iff $A \subseteq \cup \{S[y, r] : y \in B\}$. Conversely, suppose that $A \subseteq \cup \{S[y, r] : y \in B\} = B + r$. Also, $d(A, B) = \sup \{d(x, y); y \in B\}$. Let $x \in A$, then $x \in B + r$ and so, there exists some $y \in B$ such that $d(x, y) \leq r \Rightarrow d(x, B) \leq r \ \forall \ x \in A$. Hence, $d(A, B) \leq r$. □

Proposition 3.24. Let $A, B \in$ comp (X) and $r \geq 0$ be a real number. Then $d^L(A, B) \leq r$ iff $A \subseteq B + r$ and $B \subseteq A + r$.

Proof. Suppose $d^L(A, B) \leq r$. Then $d(A, B) \leq r$ and $d(B, A) \leq r$. And by 3.23, $A \subseteq B + r$ and $B \subseteq A + r$. Conversely, if $A \subseteq B + r$ and $B \subseteq A + r$ then by 3.23, we have $d(A, B) \leq r$ and $d(B, A) \leq r \Rightarrow d^L(A, B) \leq r$. □

The following result follows from the above proposition

Proposition 3.25. Let $\{A_n\}$ be a sequence in comp (X). Then $\{A_n\}$ is Cauchyif and only if for a given $\in > 0$ there exists a positive integer n_0 such that $A_n \subseteq A_m + \in$ and $A_m \subseteq A_n + \in \forall \ n, m \geq n_0$.

Proof. Suppose $\{A_n\}$ is a Cauchy sequence in comp (X). Then for a given $\in > 0$, there exists a positive integer n_0 such that $d^L(A_n, A_m) \leq \in \forall \ n, m \geq n_0$. By 3.24, we then have $A_n \subseteq A_m + \in$ and $A_m \subseteq A_n + \in \forall \ n, m \geq n_0$. Conversely, if $A_n \subseteq A_m + \in$ and $A_m \subseteq A_n + \in \forall \ n$,

$m \geq n_0$ then again by 3.24, we have $d^L(A_n, A_m) \leq \in \forall\ n, m \geq n_0$ and hence $\{A_n\}$ is a Cauchy sequence.□

Lemma 3.26. Let $\{A_n\}$ be a sequence of points of comp (X). Let $\{n_i\}$ be strictly increasing sequence in *IN*. For each $i \in IN$, let $x_{n_i} \in A_{n_i}$. Let $j \in IN$. Then for each m such that $n_{j-1} + 1 \leq m < n_j$, we can find $z_m \in A_m$ such that $d(z_m, x_{n_j}) \leq d(A_m, A_{n_j})$.

Proof. Consider the set $B = \{x \in A_m : d(x, x_{n_j}) = d(x_{n_j}, A_m)\}$. Since A_m is compact for each $m \in IN$, by 3.8, $B \neq \varnothing$. Let $z_m \in B$ be any element. Then $d(z_m, x_{n_j}) \leq d(x_{n_j}, A_m) \leq d(A_m, A_{n_j})$ by 3.11 (i). Hence the result stands proved.□

Proposition 3.27. Let X be a metric space and let $\{A_n\}$ be a Cauchy sequence of points of comp (X). Let $\{n_j\}$ be a strictly increasing sequence of natural numbers. Suppose $\{x_{n_j} \in A_{n_j}\}$ be a Cauchy sequence in X. Then there exists a Cauchy sequence $\{y_n \in A_n\}$ such that $y_{n_j} = x_{n_j}\ \forall\ j$.

Proof. Let $n \in IN$. If $n = n_i$, for some $i \in IN$, then we take $y_n = x_{n_j} \forall\ i \in IN$. We can find $j \in IN$ such that $n_{j-1} + 1 \leq m < n_j$, by 3.26, $\exists\ z_n \in A_n$ such that

$$d(z_n, x_{n_j}) \leq d(A_n, A_{n_j}) \quad \ldots (1).$$

Taking $y_n = z_n$, we clearly have $y_{n_j} = x_{n_j}$. Next, we will show that $\{y_n\}$ is a Cauchy sequence. Let $\in > 0$ be given. Since $\{x_{n_j}\}$ is a Cauchy sequence, $\therefore\ \exists\ a\ N_1 \in IN$ such that

$$D(x_{n_k}, x_{n_t}) < \in/3\ \forall\ n_k, n_t \geq N_1 \quad \ldots (2)$$

Since $\{A_n\}$ is a Cauchy sequence, therefore for taken $\in > 0\ \exists\ N_2 \in IN$ such that

$$d(A_m, A_n) < \in/3\ \forall\ m, n \geq N_2 \quad \ldots (3)$$

Let $N = N_1 \vee N_2$. Let $m, n \geq N$, then $\exists\ j, k \in IN$ such that $n_{j-1} + 1 \leq m < n_j$, and also $n_{k-1} \leq m \leq n_k$, By 3.26, $d(y_n, x_{n_j}) \leq d(A_n, A_{n_j})$ and $d(y_m, x_{n_k}) \leq d(A_m, A_{n_k})$. We have

$$n_j, n_k > N \quad \ldots (4).$$

Now, $d(y_m, y_n) \leq d(y_m, x_{n_k}) + d(x_{n_k}, x_{n_j}) + d(x_{n_j}, y_n) < 3. \in/3 = \in$ by (1), (2), (3), and (4). Hence, $\{y_n \in A_n\}$ is a Cauchy sequence. □

Theorem 3.28. Let X be a complete metric space and let $\{A_n\}$ be a Cauchy sequence of points of comp (X). Then $\{A_n\}$ converges to $A \in$ comp (X), where $A = \{x \in X : \exists\ a$ Cauchy sequence $\{x_n \in A_n\}$ that converges to $x\}$ and so comp (X) is a complete metric space.

Proof. In order to prove the theorem we shall prove that

(i) $A \neq \varnothing$

(ii) A is closed and hence complete

(iii) For $\in > 0\ \exists\ a$ natural number N such that for $n \geq N$, $A \subseteq A_n + \in$

(iv) A is totally bounded and thus compact

(v) $\lim_{n \to \infty} A_n = A$.

Now,

(i) Since $\{A_n\}$ is a Cauchy sequence, for $\in = \frac{1}{2}$ there exists a natural number n_1 such that $d^L(A_n, A_m) < \frac{1}{2}$ $\forall\, n, m \geq n_1$. For $\in = \frac{1}{2^2}$ there exists a natural number n_2 such that $d^L(A_n, A_m) < \frac{1}{2^2}$ and so on. Thus we get a strictly increasing sequence $\{n_i\}$ of natural numbers such that

$$d^L(A_n, A_m) < \frac{1}{2^i} \ \forall\, n, m \geq n_i \qquad \ldots (1).$$

Let $x_{n_1} \in A_{n_1}$, then by 3.11 (i) and (1), $d(x_{n_1}, A_{n_2}) \leq d(A_{n_1}, A_{n_2}) \leq d^L(A_{n_1}, A_{n_2}) < \frac{1}{2}$. By 3.8, there exists $x_{n_2} \in A_{n_2}$ such that $d(x_{n_1}, x_{n_2}) = d(x_{n_1}, A_{n_2})$ and therefore, we have $d(x_{n_1}, x_{n_2}) < \frac{1}{2}$.

Suppose, we have chosen a finite sequence $\{x_{n_i}\}$, $1 \leq i \leq k$, for which $d(x_{n_{i-1}}, x_{n_i}) < \frac{1}{2^{i-1}}$. Again by (1), $d^L(A_{n_k}, A_{n_{k+1}}) \leq \frac{1}{2^k}$. Since $x_{n_k} \in A_{n_k}$, as before we can find $x_{n_{k+1}} \in A_{n_{k+1}}$ such that $d(x_{n_k}, x_{n_{k+1}}) \leq \frac{1}{2^k}$. Thus we have a sequence $\{x_{n_i}\}$ such that $d(x_{n_i}, x_{n_{i+1}}) \leq \frac{1}{2^i}$. Since $\sum_{i=1}^{\infty} \frac{1}{2^i}$ is convergent, for $\in > 0$ we can find $n(\in)$ such that $\sum_{i=1}^{\infty} \frac{1}{2^i} < \in$, $n(\in) \leq I \leq \infty$. This implies that $\{x_{n_i}\}$ is a Cauchy sequence. Therefore there exists a Cauchy sequence $\{a_i \in A_i\}$ for which $a_{n_i} = x_{n_i}$. As X is complete, $\{a_i\}$ converges in X. Let $\lim a_i = a_0$. By definition of A, $a_0 \in A$, and hence, $A \neq \varnothing$.

(ii) Suppose a_i is a sequence of points of A such that $a_i \to a$. Let $i \in IN$. There exists a sequence, by definition of A, $x_{i,n} \in A_i$ such that

$$x_{i,n} \to a_i \qquad \ldots (2)$$

Since $a_i \to a$, therefore $\exists$ a natural number n_1 such that $d(a_{n_1}, a) < 1$. Similarly, $\exists\, n_2 > n_1$ such that $d(a_{n_2}, a) < \frac{1}{2}$ and so on. Therefore, we have a strictly increasing sequence $\{n_i\}$ of natural numbers s.t.

$$d(a_{n_i}, a) < \frac{1}{i} \qquad \ldots (3)$$

For each n_i, by (2), $\exists$ an integer m_i such that

$$d(x_{n_i, m_i}, a_{n_i}) < \frac{1}{i} \qquad \ldots (4)$$

Using (3) and (4) we have,

$$d\,(x_{n_i,m_i}, a) < \frac{2}{i} \qquad \ldots (*).$$

Let $y_{n_i} = x_{n_i,m_i}$. Then $y_{n_i} \in A_{n_i}$ and $\lim_{i\to\infty} y_{n_i} =$. Therefore by 3.27, $\{y_{n_i}\}$ can be extended to a convergent sequence, say $\{z_i \in A_i\}$ such that $z_{n_i} = y_{n_i}$ and $z_{n_i} \to a$. Thus $a \in A$ and hence A is closed. Also, A being closed is complete as X is complete.

(iii) Since $\{A_n\}$ is a sequence of points of comp (X), therefore for $\in > 0$, $\exists$ a natural number N such that $d^L(A_m, A_n) < \in \forall\, n, m \geq N$. Let $n \geq N$. For $m \geq n$, by 3.24,

$$A_m \subseteq A_n + \in \qquad \ldots (5)$$

Let $a \in A$. Then there exists a sequence say $\{a_i \in A_i\}$ which converges to a. For given $\in > 0$, suppose N is the positive integer, the existence of such an N is ensured if we take N to be large enough, such that for $m \geq N$, $d(a_m, a) < \in$. By (5), $a_m \in A_n + \in$. By 3.21, $A_n + \in$ is closed and so for $m \geq N$, $a \in A_n + \in$ and hence $A \subseteq A_n + \in$.

Suppose A is not totally bounded. This implies for some $\in > 0$, we can find a sequence $\{n_i\}$ in A such that

$$d\,(x_i, x_j) \geq \in \text{ for } i \text{ other than } j \qquad \ldots (6)$$

Thus there exists n such that $A \subseteq A_n + \in/3$. This implies for every $x_i \in A$, $\exists$ some $y_i \in A_n$ such that $d(x_i, y_i) < \in/3$. Since A_n is compact, $\{y_i \in A_n\}$ has a convergent subsequence, say $\{y_{n_i}\}$. By definition of a convergent sequence, we can find two points y_{n_i} and y_{n_j} such that $d(y_{n_i}, y_{n_j}) < \in/3$. We thus have, $d(x_{n_i}, x_{n_j}) \leq d(x_{n_i}, y_{n_i}) + d(y_{n_i}, y_{n_j}) + d(y_{n_j}, x_{n_j}) < 3.\in/3 = \in$, which is a contradiction to (6). Hence A is totally bounded. By (ii), A being complete is bounded.

By (iv), we have, $A \in$ comp (X). As $\{A_n\} \in$ comp (X) is a Cauchy sequence, so for a given $\in > 0$, $\exists$ a natural number N such that

$$d^L(A_m, A_n) < \in/2 \;\forall\, n, m \geq N \qquad \ldots (7)$$

By 3.24, $A_m \subseteq A_n + \in/2$. Let $n \geq N$. Let $y \in A_n$ and so there exists a natural number $N_1 > n$ such that $d^L(A_n, A_{N_1}) < \in/2$. This implies

$$A_n \subseteq A_{N_1} + \in/2 \qquad \ldots (8)$$

Similarly, there exists $N_2 > N_1 > n$ such that $d^L(A_{N_1}, A_{N_2}) < \frac{\in}{2^2}$ and so on. Thus, we have a strictly increasing sequence $\{n_i\}$ of natural numbers. Let $m, k \geq N_j$. Then $d^L(A_m, A_k) < \frac{\in}{2^{j+1}}$.

Since $y \in A_n$, $d(y, A_{N_1}) \leq d(A_n, A_{N_1})$ by 3.11 (i). By 3.17, $d(A_n, A_{N_1}) \leq d^L(A_n, A_{N_1})$. Therefore, $d(y, A_{N_1}) < \in/2$. Thus for $y \in A_n$ there exists an $x_{N_1} \in A_{N_1}$ such that $d(y, x_{N_1}) < \in/2$. Repeating the arguments we have a sequence $\{x_{N_i}\}$ of points of $\{A_{N_i}\}$ such that

$$d\,(x_{N_j}, x_{N_{j+1}}) < \frac{\in}{2^{j+1}} \qquad \ldots (9).$$

Let $j \in IN$. We have,

$D\,(y, x_{n_j}) \leq d(y, x_{n_1}) + d(x_{n_1}, x_{n_2}) +$ ---- $+ d(x_{n_{j-1}}, x_{n_j}) < \frac{\in}{2^1} + \frac{\in}{2^2} + \ldots + \frac{\in}{2^j} < \in$ and thus,

$$D\,(y, x_{n_j}) < \in \qquad \dots (10)$$

By (9), $\{x_{N_i}\}$ is a Cauchy sequence of points of $\{A_{N_i}\}$ which is complete being compact, and therefore x_{N_i} converges to a point say x. So, by definition of A, $x \in A$. By (10), $d\,(y, x_{n_j}) < \in \Rightarrow d\,(y, x) < \in \Rightarrow A_n \subseteq A + \in$ for $n \geq N$. By (iii), $A \subseteq A_n + \in$ for $n \geq N$. Thus, we have $A \subseteq A_n + \in$ and $A_n \subseteq A + \in$ for $n \geq N$. Therefore, by 3.24, $d^L\,(A_n, A) < \in$ for $n \geq N$ $\Rightarrow A_n \to A$ in comp (X). Hence comp (X) is a complete metric space. □

The space (comp (X), d^L) is known as a space of FRACTALS

Definition. Let X be a metric space. Let $f : X \to X$ be a map. The forward iterations of f are functions $f^{0n} : X \to X$ defined by $f^{00}\,(x) = x$, $f^{01}\,(x) = f\,(x)$, … $f^{0(n+1)}\,(x) = f \, o \, f^{0n}\,(x)$ $= f\,(f^{0n}\,(x))$ for $n = 0, 1, 2, 3, \dots$ and if f is invertible then the backward iterations of f are functions defined as : $f^{0(-m)} : X \to X$ defined by $f^{0(-1)}\,(x) = f^{-1}\,(x)$, … $f^{0(-m)}\,(x) = (f^{0m})^{-1}\,(x)$ for $m \in IN$.

Proposition 3.29. Let X be a metric space and let $w : X \to X$ be a continuous map. Then w maps comp (X) into itself.

Proof. Let A be a non empty subset of comp (X). This implies $w\,(A) = \{w\,(x) : x \in A\}$ $\neq \varnothing$. We shall prove that $w\,(A)$ is compact. Let $\{y_n = w\,(x_n)\}$ be an infinite sequence of points of $w\,(A)$. We have $\{x_n\}$ is a sequence of points of A. Since A is compact, the sequence $\{x_n\}$ has a convergent subsequence, say $\{x_{n_i}\}$ converging to some point, say $x_0 \in A$. Now, since w is continuous, therefore $\lim y_{n_i} = \lim w\,(x_{n_i}) = w\,(\lim x_{n_i}) = w\,(x_0) = y_0$ say. Hence, $w\,(A)$ is compact. □

Lemma 3.30. Let X be a metric space and let $w : X \to X$ be a contraction mapping with contractivity factor 's' then, w : comp (x) $\to$ comp (x) defined by $w\,(B) = \{w\,(x) : x \in B\}\ \forall\ B$ $\in$ comp (x) is a contraction map on comp (x) with contractivity factor 's'.

Proof. Let $B, K \in$ comp (x). Then $d\,(w\,(B), w\,(K)) = \sup\,\{d\,(w\,(x), w\,(K)) : x \in B\}$ $= \sup\,\{\inf\,\{d\,(w\,(x), w\,(y)\} : y \in K\} : x \in B\} \leq \sup\,\{\inf\,\{s\,d\,(x, y) : \in K\} : x \in B\}$ $\leq \sup\,\{\inf\,\{s\,d\,(\underline{x}, K) : x \in B\} = s\,d\,(B, K)$. Thus, $d\,(w\,(B), w\,(K)) \leq s\,d\,(B, K)$. Similarly, $d\,(w\,(K), w\,(B)) \leq s\,d\,(K, B)$. Therefore, $d^L\,(w\,(B), w\,(K)) = s\,d\,(w\,(B), w\,(K)) \vee d\,(w\,(K), w\,(B)) \leq s\,[d\,(B, K) \vee d\,(K, B)] \leq s\,d^L\,(B, K)$. □

Lemma 3.31. Let X be a metric space. Let $\{w_n : n \in J_N\}$ be contraction mappings on comp (X). s_n be the contractivity factor for w_n for each n. Then the map w : comp (x) $\to$ comp (x) defined by $w\,(B) = w_1\,(B) \cup w_2\,(B) \cup w_3\,(B)\ \dots \cup w_n\,(B) = \bigcup_{n=1}^{N} w_n(B)$ for each $B \in$ comp (x) is a contraction map on comp (x) with contractivity factor $s = \sup\,\{s_n : n \in J_N\}$.

Proof. Let $B, K \in$ comp (x). We have, for $n = 2$, $d^L\,(w\,(B), w\,(K)) = d^L\,(w_1\,(B) \cup w_2\,(B), w_1\,(K) \cup w_2\,(K)) \leq d^L\,(w_1\,(B), w_1\,(K)) \vee d^L\,(w_2\,(B), w_2\,(K)) \leq s_1 d^L\,(B, K) \vee s_2 d^L\,(B, K)$ $\leq s\,d^L\,(B, K)$ where $s = s_1 \vee s_2$. Hence, the result is valid for $n = 2$. Similarly the result holds for each $n \in J_N$. □

Definition. An iterated function system, abbreviated as IFS, consists of a complete metric space X together with a finite set of contraction mappings $w_n : X \to X$ with s_n, $n \in J_N$ as respective contractivity factors.

Notation. Let X be a complete metric space. Then IFS is $\{x : w_n, n \in J_N\}$ and its contractivity factor is $s = \sup\,\{s_n : n \in J_N\}$.

Theorem 3.32. Let $\{x : w_n, n \in J_N\}$ be an IFS with contractivity factor 's'. Then the function w : comp $(x) \to$ comp (x) defined by $w(B) = \cup\, w_n(B)$, $1 \le n \le N$, for all $B \in$ comp (x), is a contraction mapping on comp (x) with contractivity factor 's' that is $d^L(w(B), w(K)) \le s\, d^L(B, K)$ $\forall$ $B, K \in$ comp (x), and the unique point $A \in$ comp (x) of w is such that $A = w(A) = \cup\, w_n(A)$, $1 \le n \le N$, and is given by $A = \lim_{n \to \infty} w(B)^{0n}$ for any $B \in$ comp (x).

Proof. Let X be a complete metric space. Let w_n, $n \in J_N$ be contractions on X with s_n, $n \in J_N$ as respective contractivity factors. Let w : comp $(x) \to$ comp (x) be defined as $w(B) = \cup\, w_n(B)$, $1 \le n \le N$, for all $B \in$ comp (x). By 3, 31, w is a contraction on comp (X) with 's' as contractivity factor, where $s = \sup \{s_n : n \in J_N\}$. Hence, $d^L(w(B), w(K)) \le s\, d^L(B, K)$ $\forall$ $B, K \in$ comp (x). Therefore, w has a unique fixed point, say A, where $A \in$ comp (x) and also $\lim_{n \to \infty} w(B)^{0n} = A$ for any $B \in$ comp (x). Hence the result is proved. □

Definition 3.33. The fixed point $A \in$ comp (x) as described in 3.32 is called the attractor of the IFS. A fixed point of a contraction mapping on (comp (X), d^L) is defined as a *deterministic fractal*.

Proposition 3.34. Let C_d be a code space on K_3. Then $\{C_d : w_n, n = 1, 2\}$ is an IFS where $w_1, w_2 : C_d \to C_d$ are contraction mappings defined suitably.

Proof. We know that the Code Space, C_d, on 3 symbols, *i.e.*, for $N = 3$ is a complete metric space. We, therefore, define w_1 and w_2 on C_d as follows : Let $x = x_1\, x_2\, x_3 \ldots \in C_d$ and define $w_1, w_2 : C_d \to C_d$ as $w_1(x = x_1\, x_2\, x_3 \ldots) = 0\, x_1\, x_2\, x_3 \ldots$. and $w_2(x = x_1\, x_2\, x_3 \ldots) = 2x_1\, x_2\, x_3 \ldots$. Then we know that w_1, w_2 are contraction mappings on C_d with contractivity factor 1/3. $\{C_d : w_n, n = 1, 2\}$ is an IFS and its contractivity factor 1/3. □

Remark 3.35. It is clear from the definition of attractor of an IFS, that every attractor is a deterministic fractal.

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