Acta Ciencia Indica, Vol. XLII M, No. 1, 35 (2016)

IDEALS OF A RING BASED ON ROUGH SETS

ARVIND KUMAR SINHA AND ANAND PRAKASH

Department of Mathematics, National Institute of Technology, Raipur, C.G., India

RECEIVED : 27 February, 2016

REVISED : 22 March, 2016

The aim of this paper is to study ideals and prime ideals based on rough set theory and characterize them.

KEYWORDS : Rough sets, Rough Ideals, Rough Ring, Rough modules.

INTRODUCTION

Z. Pawlak ([10], in 1982), proposed the concept of rough set theory from then, there has been a fast growing interest in this field including pure theory, algebraic foundations and application can be found in [5, 8, 10, 12, 16, 17, 19]. Pawlak gives the algebraic foundation of rough sets in [8], then lot of work is done in this field. Some of related work is cited here. In [1], Q. Xiao and Z. Zhang discusses the rough prime ideals and rough fuzzy prime ideals. In [2], fuzzy ideals of a ring have been discussed. in [3], rough free module is discussed. In [4] fuzzy prime ideals of a ring are discussed. B. Davvaz gives roughness based on fuzzy ideals in [6] and in [9], roughness in rings. Kuroki [7] gives rough ideals in semigroups. Algebraic ideals are given in [11, 18]. In recent article [13, 14, 15, 20] are important in the study of rough ideals.

We have used standard mathematical notations throughout this paper and we assume that reader is familiar with basic notions of rough set theory and algebra.

PRELIMINARIES

Let U be an universal set, let θ an equivalence relation on U then the equivalence class of x, is the set of the elements of U that are related to $x \in U$, and denoted as $[x]_\theta$. A pair (U, θ), where $U \neq \emptyset$ and θ is an equivalence relation on U is called an approximation space. By a rough approximation in (U, θ) we mean a mapping $Apr: P(U) \to P(U) \times P(U)$ defined for every $x \in P(U)$ by

$$Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$$

where, $\underline{Apr}(X) = \{x \in U | [x]_\theta \subseteq X\}$, $\overline{Apr}(X) = \{x \in U | [x]_\theta \cap X \neq \emptyset\}$. $\underline{Apr}(X)$ and $\overline{Apr}(X)$ are called lower and upper approximation of X in (U, θ), respectively. It is easy to see that congruent relation is an equivalent relation. In this paper we take R as a ring under usual operation. I be an ideal of R, and let $X(\neq \emptyset) \subseteq R$, then define $\underline{Apr_I}(X) = \{x \in U | (x + I) \subseteq X\}$, $\overline{Apr_I}(X) = \{x \in U | (x + I) \cap X \neq \emptyset\}$, are called the lower and upper approximation of the set X with respect to ideal I, respectively. Some time, We also use notations

$$\rho_-(A) = \{x \in R | [x]_\rho \subseteq A\}$$

174/M016

$$\rho^{-}(A) = \{x \in R | [x]_{\rho} \cap A \neq \emptyset\}$$

as a lower and upper approximation of a set A, a subset or ring R and ρ a congruence relation on R. $\rho(A) = (\rho_{-}(A), \rho^{-}(A))$ is called a rough set with respect to ρ if $\rho^{-}(A) \neq \rho_{-}(A)$. A congruence relation on R is called complete if $[a]_{\rho}[b]_{\rho} = [ab]_{\rho}$ for any $a, b \in R$.

Proposition 1 [9]. Let I be an ideal of R, and A, B nonempty subsets of R, then $\overline{Apr}_I(A) \cdot \overline{Apr}_I(B) = \overline{Apr}_I(A \cdot B)$.

Proposition 2 [9]. Let I be an ideal of R, and A, B nonempty subsets of R, then $\underline{Apr}_I(A) \cdot \underline{Apr}_I(B) = \underline{Apr}_I(A \cdot B)$.

IDEALS BASED ON ROUGH SET

Theorem 3.1. Let ρ and λ be a congruence relation on a ring R. If A and B are nonempty subsets of S, then $\rho_{-}(A) \subseteq A \subseteq \rho^{-}(A)$ and $\rho^{-}(A \cup B) = \rho^{-}(A) \cup \rho^{-}(B)$.

Proof. if $a \in \rho_{-}(A)$, then $a \in [a]_{\rho} \subseteq A$ implies $\rho_{-}(A) \subseteq A$. Now, if $a \in A$, then we have $[a]_{\rho} \cap A \neq \emptyset$, this implies that $a \in \rho^{-}(A)$ implies that $A \subseteq \rho^{-}(A)$. For second part let

$$\begin{aligned} a \in \rho^{-}(A \cup B) &\Leftrightarrow [a]_{\rho} \cap (A \cup B) \neq \emptyset \\ &\Leftrightarrow ([a]_{\rho} \cap A) \cup ([a]_{\rho} \cap B) \neq \emptyset \\ &\Leftrightarrow [a]_{\rho} \cap A \neq \emptyset \quad or \quad [a]_{\rho} \cap B \neq \emptyset \\ &\Leftrightarrow a \in \rho^{-}(A) \quad ora \in \rho^{-}(B) \\ &\Leftrightarrow a \in \rho^{-}(A) \cup \rho^{-}(B). \end{aligned}$$

This completes the proof.

Definition 3.2. Let ρ is a congruence relation on R, R is a ring in usual manner. Let A a subset of R. A rough set $\rho(A) = (\rho_{-}(A), \rho^{-}(A))$ is a rough ideal of R, if $\rho_{-}(A)$ and $\rho^{-}(A)$ are ideals of R.

Example 3.3. Let $R = \mathbb{Z}_{12}$, $I = \{0,4\}$ an ideal of R. Let $A = \{0,4,6\}$ a subset of R. Then $I_{-}(A) = \{0,4\}$ and $I^{-}(A) = \{0, 2, 4, 6\}$ are the lower and upper approximation of A with respect to a congruence relation I on R. Since $I_{-}(A)$ and $I^{-}(A)$ are ideals in R. This implies that $I(A) = (I_{-}(A), I^{-}(A))$ is a rough ideal.

Definition 3.4. A ring R is said to be regular , if for each $a \in R$ there exists $x \in R$ such that $a = axa$.

Theorem 3.5. Let ρ be a congruence relation on a ring R. If A and B are nonempty subsets of R, then $\rho^{-}(A)\rho^{-}(B) \subseteq \rho^{-}(AB)$.

Proof. Let c is an element of $\rho^{-}(A)\rho^{-}(B)$ implies $c = ab$ with $a \in \rho^{-}(A)$ and $b \in \rho^{-}(B)$, then there exists elements $r, s \in R$ such that $r \in [a]_{\rho} \cap A$ and $s \in [a]_{\rho} \cap B$. This implies that $r \in [a]_{\rho}$ and $s \in [a]_{\rho}$, $r \in A, s \in B$, Since ρ is a congruence relation in R, $rs \in [a]_{\rho}[b]_{\rho} \subseteq [ab]_{\rho}$. Since $rs \in AB$, we have $rs \in [ab]_{\rho} \cap AB$ ans $ab \in \rho^{-}(AB)$. This completes the proof.

Theorem 3.6 Let ρ be a congruence relation on a ring R. If A is a subring of R, then A is an upper rough subring of R. If A is an ideal of R, then A is an upper rough ideal of R.

Proof. Let A be a subring of R, then we have $\emptyset \neq A \subseteq \rho^{-}(A)$. Then by theorem 3.5 we have $\rho^{-}(A)\rho^{-}(A) \subseteq \rho^{-}(AA) \subseteq \rho^{-}(A)$. This implies that $\rho^{-}(A)$ is a subring of R, this completes first part. For second part let A in an ideal of R, that is , $RA \subseteq A$. Since $\rho^{-}(R) = R$. Then using theorem 3.5, we have $R\rho^{-}(A) = \rho^{-}(R)\rho^{-}(A) \subseteq \rho^{-}(RA) \subseteq \rho^{-}(A)$, this means that $\rho^{-}(A)$ is an ideal of R. Hence A is an upper rough ideal of R. similarly right ideal can be applied. This completes the proof.

A subring S of a ring R is called a bi-ideal of R if $SRS \subseteq S$. A subset S of R is called a ρ-upper rough bi ideal of R if $\rho^{-}(S)$ is a bi-ideal of R.

Theorem 3.7. Let ρ be congruence relation on a ring R. If A be a right ideal and B a left ideal of R then $\rho^{-}(AB) \subseteq \rho^{-}(A) \cap \rho^{-}(B)$.

Proof. Given that A is a right ideal of R implies $AB \subseteq AR \subseteq A$ and given that B is a left ideal R implies $AB \subseteq RB \subseteq B$ this implies that $AB \subset of A \cap B$, then it gives that $\rho^{-}(AB) \subseteq \rho^{-}(A \cap B) \subseteq \rho^{-}(A) \cap \rho^{-}(B)$. This completes the proof.

Theorem 3.8. Let ρ be a complete congruence relation on a ring R. Let A be an ideal of R, then $\rho_{-}(A)$ is an ideal of R, if it is nonempty.

Proof. Let A be an ideal of R, that is $RA \subseteq A$ and $AR \subseteq A$. Note that $\rho_{-}(R) = R$.Then by proposition 2, we have

$$R\rho_{-}(A) = \rho_{-}(R)\rho_{-}(A) \subseteq \rho_{-}(RA) \subseteq \rho_{-}(A)$$

This means that $\rho_{-}(A)$ is an ideal of R. This implies that $\rho_{-}(A)$ is an lower rough ideal of R.

PRIME IDEALS BASED ON ROUGH SET IN A RING

Definition 4.1. An ideal of a ring R is a prime ideal of R such that $xy \in A$ for some $x, y \in R$ implies $x \in A$ or $y \in A$. Let ρ be a congruence relation on a ring R. Then a subset I of R is called a lower rough prime ideal of R if $\rho_{-}(I)$ is a prime ideal of R, and upper rough prime ideal of R if $\rho^{-}(I)$ is a prime ideal of R.

Example 4.2. Let $R = \mathbb{Z}_{12}$, $I = \{0,6\}$, $A = \{0,1,2,5,6,8\}$, $B = \{0,3,4,6,9\}$. I is an ideal over ring R. A is any subset or R, obviously I is a congruence relation. We define lower approximation of A with respect to congruence relation I is $\underline{Apr_I}(A) = \{x \in R | x + I \subseteq A\}$ and upper approximation is $\overline{Apr_I}(A) = \{x \in R | (x + I) \cap A \neq \emptyset\}$.

Now classes of $I = \{2,8\},\{4,10\},\{3,9\},\{1,7\},\{6,0\},\{5,11\}$. Thus $\underline{Apr_I}(A) = \{0,2,6,8\}$, $\overline{Apr_I}(A) = \{0,1,2,5,6,7,8,11\}$ This implies that $\underline{Apr_I}(A)$ and $\overline{Apr_I}(A)$ are prime ideals of $R = \mathbb{Z}_{12}$ and this implies that $Apr(A) = (\underline{Apr_I}(A), \overline{Apr_I}(A))$, is a prime ideal of $R = \mathbb{Z}_{12}$.

Theorem 4.3. Let ρ be a complete congruence relation on a ring R and A a prime ideal of R. Then $\rho_{-}(A)$ is , if it is nonempty, a prime ideal or R.

Proof. Let A is an ideal of R, by theorem 3.8, we know that $\rho_{-}(A)$ is an ideal of R. Let $xy \in \rho_{-}(A)$ for some $x, y \in R$. Then $[x]_\rho[y]_\rho \subseteq [xy]_\rho \subseteq A$. we suppose that $[x]_\rho$ is not a prime ideal, then there exists, $x, y \in R$ such that $xy \in \rho_{-}(A)$ but $x \neq \rho_{-}(A)$ and $y \neq \rho_{-}(A)$. Thus $[x]_\rho$ÚA and $[y]_\rho$ÚA, then exists $x' \in [x]_\rho$, $x' \notin A$ and $y' \in [y]_\rho$, $y' \notin A$. Thus $x'y' \in [x]_\rho[y]_\rho \subseteq A$. Since A is prime ideal, we have $x' \in A$ or $y' \in A$. THis contradicts the supposition. This means that $\rho_{-}(A)$ is, if it is non-empty, a prime ideal of R.

Theorem 4.4. Let ρ be a complete congruence relation on a ring R. If I is a prime ideal of R, then I is an upper rough prime ideal of R.

Proof. Since I is a prime ideal of ring R, by theorem ?? we know that $\rho^{-}(I)$ is an ideal of R. Let $xy \in \rho^{-}(I)$ for some $x, y \in R$, then $[xy]_\rho \cap I = [x]_\rho[y]_\rho \cap I \neq \emptyset$. So there exists $x' \in [x]_\rho$ and $y' \in [y]_\rho$ such that $x'y' \in I$, since I is a prime ideal, we have $x' \in I$ or $y' \in I$. Thus $[x]_\rho \cap I \neq \emptyset$ or $[y]_\rho \cap I \neq \emptyset$, and so $[x]_\rho \cap I \neq \emptyset$ or $[y]_\rho \cap I \neq \emptyset$. Therefore $\rho^{-}(I)$ is a prime ideal of R. The converse of this theorem does not hold in general.

Conclusion

The theory of rough sets is mathematical tool to deal with vagueness. In this paper we discussed ideals and prime ideals based of rough set theory. We hope these results will further enrich mathematical foundation of rough set theory.

References

1. Xiao, Q. and Zhang, Z., Rough prime ideals and rough fuzzy prime ideals in semigroups, *Information Sciences*, **176,** 725-733 (2006). DOI:10.1016/j.ins.2004.12.010.
2. Mukherjee, T.K. and Sen, M.K., On Fuzzy Ideals of a Ring I, *Fuzzy Sets and Systems*, **21,** 99-104 (1987). DOI:10.1016/0165-0114(87)90155-2
3. Sinha, A. K. and Prakash, Anand, Roughness in Free Module, *The first int. conference on Rough Sets & Knowledge Technologies India*, Page 19-22, 10-12 November (2014).
4. Malik, D.S. and Mordeson, John N., Fuzzy Prime Ideals of a Ring, *Fuzzy Sets and Systems*, **37,** 93-98 (1990). DOI: 10.1016/0165-0114(90)90066-F
5. Ali, M.I., Davvaz, B. and Shabir, M., Some properties of generalized rough sets, *Information Sciences*, **224**, 170-179 (2013). DOI : 10.1016/j.ins.2012.10.026.
6. Davvaj, B., Roughness based on fuzzy ideals, *Information Sciences*, **176,** 2417-2437 (2006). DOI:10.1016/j.ins.2005.10.001.
7. Kuroki, N., Rough ideals in semigroups, *Infor. Sci*., **100**, 136-163 (1997). DOI : 10.1016/S0020-0255(96)00274-5
8. Pawlak, Z., *Rough sets: Theoretic aspects of reasoning about data*, Boston, Kluwer Academic Publicators (1991).
9. Davvaz, B., Roughness in rings, *Infor. Sci*., **164**, 147-163 (2004). DOI : 10.1016/j.ins.2003.10.001.
10. Pawlak, Z., Rough sets, *Int. J. Inf. Comput. Sci*., **11**, 341-356 (1982). DOI: 10.1007/BF01001956.
11. Anderson, F. W. and Fuller, K. R., *Rings and Categories of Modules*, **Second Edition**, Springer-Verlag (1992).
12. Sinha, A.K. and Prakash, Anand, Injective module based on rough set theory, *Cogent Mathematics*, **2** (2015). DOI: 10.1080/23311835.2015.1069481
13. Zhan, J. and Davvaz, B., Notes on roughness in rings, *Infor. Sci*., (2015).
14. Prakash, Anand and Sinha, A.K., Rough Projective Module, *Int. Conf. on Advances in Applied Science And Environmental Engineering* – ASEE, pp 35-38 (2014). DOI: 10.15224/ 978-1-63248-033-0-09
15. Davvaz, B. and Mahdavipour, M., Roughness in modules, *Infor. Sci*., 176, 3658-3674 (2006). DOI: 10.1016/j.ins.2006.02.014
16. Biswas, R. and Nanda, S., Rough groups and rough subgroups, *Bull. Polish Acad. Sci. Math*, **42**, 251-154 (1994).
17. Suqing, Han, The homomorphism and isomorphism of rough groups, *Acad of Shanxi University*, **24**, 303-305 (2001).
18. Luthar, L.S., Passi, I.B.S., *Algebra* Volume **3** : *Modules*, Narosha Publishing House (2002).
19. Peters, G., Lingras, P., Slezak, D. and Yao, Y., Rough sets : Selected Methods and Applications in Management and Engineering, *Springer* (2014).
20. Miao, D., Han, S., Li, D., Sun, L., Rough Group, Rough Subgroup and Their Properties, *LNAI*, 3641, pp. 104-113 (2005).

❑