Acta Ciencia Indica, Vol. XLII M, No. 1, 09 (2016)

EFFECT OF ELECTRIC LOAD PARAMETER ON HYDROMAGNETIC MIXED CONVECTION FLOW OF IMMISCIBLE VISCOUS LIQUIDS IN A VERTICAL CHANNEL WITH VISCOUS DISSIPATION AND JOULE HEATING : TWO LIQUIDS MODEL

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RECEIVED : 1 February, 2016

Mixed convection flow of two viscous, incompressible, electrically conducting, immiscible liquids in a vertical channel with adiabatic walls is investigated. The laminar parallel and fully developed regimes are considered. Both the liquids are considered with different densities, viscosities, electrical and thermal conductivities and occupy equal width. A uniform magnetic field is considered to be applied normal to the flow regime. The equations of momentum and energy for both the liquids are written in non-dimensional form and solved by taking into account the effects of electric load parameter, Joule heating and viscous dissipation. The transport properties of the liquids in both the regions are assumed to be constant except variation in densities with temperature. The solutions for velocity and temperature distribution are obtained analytically for each region using suitable non-slip, matching, boundary and interface conditions. The non-dimensional governing parameters affecting the velocity and temperature are discussed with the help of figures.

NOMENCLATURE

B_0 : Uniform magnetic field along –axis,

Ec : Eckert number,

Ey' : Electric circuit perpendicular to magnetic field,

$Gr\left(= Gr\dfrac{Pr}{\mu_r}\right)$: Grashof number in region-I,

g : Acceleration due to gravity,

K_{T_1} : Thermal conductivity of the fluid in region-I,

K_{T_2} : Thermal conductivity of the fluid in region-II,

$M\left(=\dfrac{\sigma_r}{\mu_r}M_1'\right)$: Magnetic parameter in region-I,

P' : Constant pressure gradient in region-I,

P_1' : Constant pressure gradient in region-II,

$P\left(=\dfrac{P_1}{\mu_r}\right)$: Constant pressure gradient in region-I,

Pr : Prandtl number,

T_1' : Temperature of the fluid in region-I,

T_2' : Temperature of the fluid in region-II,

u_1' : Velocity field in region-I,

u_2' : Velocity field in region-II,

GREEK SYMBOLS

β : Volumetric coefficient of expansion,

μ_1 : Viscosity of the fluid in region-I,

μ_2 : Viscosity of the fluid in region-II,

ρ_1 : Density of the fluid in region-I,

ρ_2 : Density of the fluid in region-II,

σ_1 : Electrical conductivity in region-I,

σ_2 : Electrical conductivity in region-II.

Introduction

Magnetohydrodynamics deals with the motion of electrically conducting fluids under the influence of magnetic field. Considerable attention has been given to magnetohydrodynamic (MHD) flow since of the last century. The birth date of MHD may be identified with the experiments by Faraday, who attempted to measure the electric potential induced between the opposite banks of the Thames river by the motion of the (weakly) conducting water in the Earth's magnetic field. Significant applications of MHD have been reported; such as the MHD generators, MHD flow meter, MHD pump and MHD marine propulsion. Some other quite promising applications are in the field of metallurgy; such as MHD stirring of molten metal, magnetic-levitation casting and lithium cooling blanket in a nuclear fusion reactor [1]. In addition, the study of flow behaviour of two or more immiscible liquids under the influence of magnetic field is of immense importance due to there abundance use in flow sciences. Such flow occurs in solid mechanics, ground water hydrology, purification of the crude oil in petroleum industry, oil recovery through ocean wells, flow of water containing oil in packed rocks, etc, where immiscible fluids flow exists in two or more layers. Flow of immiscible liquids have been studied by several authors, including Kapur and

Shukla [2, 3], Pathak [4], Vafai and Thiyagaraja [5], Ingham *et al*. [6] Al-Hadhrami *et al*. [7], Singh *et al*. [8], Malashetty *et al.* [9], Singh and Takhar [10], Singh *et al*. [11-15] and Singh and Deka [16].

In these studies the boundary conditions considered on the walls of the channel are either uniform temperature of uniform heat flux. Indeed these studies are based on the assumption that the effect of viscous dissipation in the fluid is negligible. This assumption holds whenever the fluid has a sufficiently high thermal conductivity and a sufficiently small Prandtl number under the influence of sufficiently high wall heat flux. As a consequence, the analysis of such flows for a straight forward analytical determination of the velocity and temperature profiles have been given due importance. Analytical solutions of hydromagnetic mixed convection problems on immiscible fluid flows in vertical, inclined or horizontal channels have been the subject of several papers in the latter decades. The importance of such analytical solutions, which refer to laminar fully developed flows, relies on the chance to obtain non-trivial land marks to test the reliability of numerical codes developed for more complex geometrics or for non-parallel flows. Moreover, such analytical solutions are often an opportunity to inspect the internal consistency of the mathematical models and of the approximations adopted, as well as to develop new theoretical results. For instance, Barletta and Zanchini [17] and Umavathi and Malashetty [18] have shown the effective utility of analytical solution through a novel criterion to choose the reference temperature by adopting the Boussinesqu approximation for fully developed mixed convection flow in a vertical channel.

Due to engineering application, theoretical investigations have been devoted to the analysis of the effect of viscous dissipation and the effect of buoyancy with reference to channel flow. Becket [19], Becket and Friend [20] investigated combined natural and forced convection walls for low and higher Rayleigh number. Barletta [21, 22] examined convective flow in a vertical channel with buoyancy effect and wall heat fluxes respectively. Barletta *et al.* [23] considered the classical problem of fully developed mixed convection flow, with frictional heat generation, in a vertical channel bounded by isothermal plane walls having the same temperature. In this model, the viscous dissipation effect is taken into account and the set of governing equations is reduced to a fourth order ordinary differential equations for the velocity field, whereas Sposito and Ciofalo [24] presented an analysis, with analytical solutions, on the parallel fully developed flow of an electrically conducting fluid between plane parallel walls under the simultaneous influence of driving pressure head, buoyancy and magnetohydrodynamic (MHD) forces. In this analysis, the fluid is assumed to be internally heated and the flow modeled as one-dimensional and incompressible. Besides, the Boussinesqu approximation is invoked for the buoyancy terms. Recently, Barletta and Celli [25] examined combined free and forced flow in a vertical channel with an adiabatic wall and an isothermal wall considering laminar, parallel and fully developed regime under the influence of uniform magnetic field applied normal to the flow regime. The local balance equations are solved by taking into account the effects of Joule heating and viscous dissipation. More recently, Barletta *et al*. [26] discussed magnetohydrodynamic mixed convection flow in a vertical porous annulus surrounding the electric cable.

In the above mentioned studies, the electric load parameter is not given due importance, although it has important applications in the design of MHD generators, cross-fired accelerators, coal-fired MHD generators and aerospace technology. The aim of the present paper is to examine effect of hydromagnetic parameter on mixed convection flow of two immiscible liquids in a vertical channel with an adiabatic wall and an isothermal wall, which are kept at the same temperature. The viscous dissipation effects as well as Joule heating effect are taken into account. The results of velocity field and temperature distribution in both the flow regions are studied with help of figures. The present study is expected to be useful in understanding the heat transfer characteristics of immiscible fluid flows through parallel channel walls in the presence of magnetic and electric field.

GOVERNING EQUATIONS AND FORMULATION OF THE PROBLEM

We consider the steady laminar fully developed flow of two viscous, incompressible, electrically conducting and immiscible liquids in a vertical parallel walls channel of width $2H$. In cartesian coordinate system the x'-axis is opposite to the gravitational $\vec{g}$ and the y'–axis is perpendicular to the channel walls, which are considered to be impermeable and isothermal. The flow is laminar and parallel to the walls, so that the velocities of both the liquids are directed along the x'-axis. The left wall (at $y' = -H$) is adiabatic and the right one (at $y' = H$) is kept at the constant temperature T_0. The region-I $(-H \le y' \le 0)$ is occupied by the liquid of density ρ_1, viscosity μ_1, electrical conducting σ_1 and thermal conductivity K_{T_1}, whereas the region-II $(0 \le y' \le H)$ is occupied by the liquid of density ρ_2, viscosity μ_2, electrical conductivity σ_2 and thermal conductivity K_{T_2}. Schematic diagram and coordinate system of the problem is shown Fig.1. Moreover, the present analysis is based on the following assumptions:

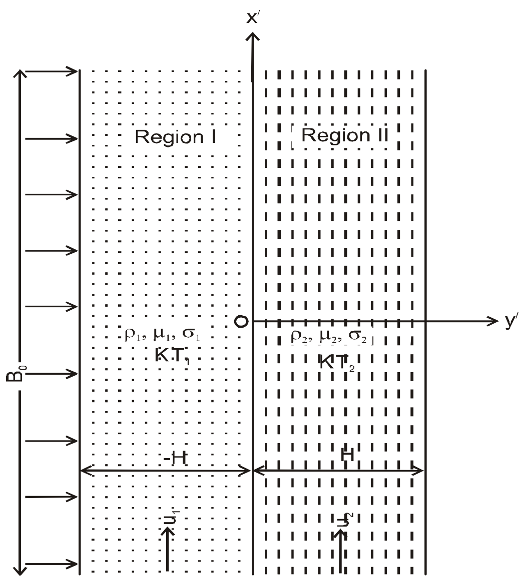

Fig. 1. Schematu diagram of the problem.

(i) The transport properties of both the liquids are constant.

(ii) The induced electric field in the region-I is $\vec{E}_1 = \vec{U}_1 \times \vec{B}$ and in the region-II is $\vec{E}_2 = \vec{U}_2 \times \vec{B}$; so that the current density is given by $\vec{J}_1 = \sigma_1 \vec{E}_1 = \sigma_1 \vec{U}_1 \times \vec{B}$ and $\vec{J}_2 = \sigma_2 \vec{E}_2 = \sigma_2 \vec{U}_2 \times \vec{B}$ respectively.

(iii) The liquid velocity and magnetic distribution are $\vec{U}_1 = (u_1'(y'), 0, 0)$, $\vec{U}_2 = (u_2'(y'), 0, 0)$ and $\vec{B} = (0, B_0, 0)$ respectively.

(iv) Since $\vec{B}$ is orthogonal to $\vec{U}_1$ and $\vec{U}_2$ the magnetic body force per unit volume in the region-I and region-II are expressed as:

` $\vec{f}_1 = \sigma_1 B_0^2 u_1'(y')$ and $\vec{f}_2 = \sigma_2 B_0^2 u_2'(y')$ respectively.

The power per unit volume generated by Joule effect in the region-I and in the region-II is:

$$q_{1_g} = \vec{J}_1.\vec{E}_1 = \sigma_1\left(\vec{U}_1 \times \vec{B}\right).\left(\vec{U}_1 \times \vec{B}\right) = \sigma_1 B^2 U_1^2$$

and $q_{2_g} = \vec{J}_2.\vec{E}_2 = \sigma_2\left(\vec{U}_2 \times \vec{B}\right).\left(\vec{U}_2 \times \vec{B}\right) = \sigma_2 B^2 U_2^2$ respectively.

(v) Since the uniform magnetic field B_0 is applied normal to walls along y'-direction, the uniform electric field is applied perpendicular to the magnetic field along – direction; as such magnetic body force reduces to:

$$\vec{f}_1 = \sigma_1 B_0\left(E_{y'} + B_0 u_1'\right) \quad \text{and} \quad \vec{f}_2 = \sigma_2 B_0\left(E_{y'} + B_0 u_1'\right) \text{ respectively.}$$

(vi) The magnetic Reynolds number is small so that induced magnetic field is neglected in comparison to applied magnetic field.

(vii) The fully developed parallel flow condition and uniform wall temperature imply that the fluid velocities $\vec{U}_1$ and $\vec{U}_2$, , and the fluid temperatures T_1' and T_2' depend only on y'.

(viii) The hydrodynamic pressures $-\dfrac{dp_1}{dx'}$ and $-\dfrac{dp_2}{dx'}$ are constant.

(ix) All the fluid properties except the density in the buoyancy term, in both regions, are considered constant.

(x) The origin is considered in the midway of the channel walls, so that both the immiscible fluids occupy equal breadth H to flow in the parallel impermeable walls.

(xi) The Boussinesqu approximation holds and that both the Joule heating and the heat generation by viscous dissipation is taken into account.

Under the present configuration, the momentum and energy equations governing the MHD flow in the region-I and the region-II can be expressed as:

Region-I $(-H \le y' \le 0)$

$$-P' + \mu_1 \frac{d^2 u_1'}{dy'^2} + g\rho_1\beta(T_1' - T_0') - \sigma_1 B_0\left(E_{y'} + B_0 u_1'\right) = 0. \qquad \ldots(1)$$

$$K_{T_1}\frac{d^2 T_1'}{dy'^2} + \mu_1\left(\frac{du_1'}{dy'}\right)^2 + \sigma_1\left(E_{y'} + B_0 u_1'\right)^2 = 0\,. \qquad \ldots(2)$$

Region-II $(0 \le y' \le H) - \dfrac{dp_2}{dx'} + \mu_2 \dfrac{d^2 u_2'}{dy'^2}$

$$-P_1' + \mu_2 \frac{d^2 u_2'}{dy'^2} + g\rho_2\beta(T_2' - T_0') - \sigma_2 B_0 \left(E_{y'} + B_0 u_2' \right) = 0\,. \qquad \ldots(3)$$

$$K_{T_2} \frac{d^2 T_2'}{dy'^2} + \mu_2 \left(\frac{du_2'}{dy'} \right)^2 + \sigma_2 \left(E_{y'} + B_0 u_2' \right)^2 = 0\,. \qquad \ldots(4)$$

The reference temperature T_0' and the temperature of the adiabatic wall are chosen equal. The non-slip conditions, matching conditions and the thermal boundary conditions relevant to the problem are given by:

$$u_1' = 0, \qquad T_1' = T_0', \qquad \text{at} \quad y' = -H.$$

$$u_1' = u_2', \qquad \mu_1 \frac{du_1'}{dy'} = \mu_2 \frac{du_2'}{dy'} \qquad \text{at} \quad y' = 0.$$

$$T_1' = T_2', \qquad K_{T_1} \frac{dT_1'}{dy'} = K_{T_2} \frac{dT_2'}{dy'} \qquad \text{at} \quad y' = 0.$$

$$u_2' = 0, \qquad T_2' = T_0' \qquad \text{at} \quad y' = H. \qquad \ldots(5)$$

We introduce following non-dimensional quantities:

$$u_1 = \frac{u_1'}{U_0}, \quad u_2 = \frac{u_2'}{U_0}, \quad y = \frac{y'}{H}, \quad T_1 = \frac{T_1' - T_0'}{T_w' - T_0'}, \quad T_2 = \frac{T_2' - T_0'}{T_w' - T_0'}.$$

Using above stated non-dimensional variables in equations (1)-(4). The equations governing the flow in non-dimensional form reduce to:

Region-I $(-1 \le y \le 0)$

$$-P + \frac{d^2 u_1}{dy^2} + GrT_1 - M^2 (E + u_1) = 0. \qquad \ldots(6)$$

$$\frac{d^2 T_1}{dy^2} + EcPr \left(\frac{du_1}{dy} \right)^2 + M^2 EcPr (E + u_1)^2 = 0. \qquad \ldots(7)$$

Region-II $(0 \le y \le 1)$

$$-P + \frac{d^2 u_2}{dy^2} + Gr_1 T_2 - M_1^2 (E + u_2) = 0. \qquad \ldots(8)$$

$$\frac{d^2 T_2}{dy^2} + EcPr \frac{K_{T_r}}{\mu_r} \left(\frac{du_2}{dy} \right)^2 + M^2 EcPr \frac{K_{T_r}}{\sigma_r} (E + u_2)^2 = 0, \qquad \ldots(9)$$

where $\quad Gr_1 = \dfrac{\mu_r}{\rho_r} Gr, \quad M_1^2 = \dfrac{\mu_r}{\sigma_r} M^2, \quad P_1 = \mu_r P.$

$P = \dfrac{P' H^2}{\mu_1 U_0}$ (constant pressure gradient in region-I), $Gr = \dfrac{g \rho_1 \beta H^2 (T'_w - T'_0)}{\mu_1 U_0}$ (Grashof number), $M^2 = B_0^2 H^2 \dfrac{\sigma_1}{\mu_1}$ (magnetic parameter), $E = \dfrac{E_{y'}}{U_0 B_0}$ (electric load parameter), $Pr = \dfrac{\mu_1 C_p}{K_{T_1}}$ (Prandtl number), $Ec = \dfrac{U_0^2}{C_p (T'_w - T'_0)}$ (Eckert number), $\mu_r = \dfrac{\mu_1}{\mu_2}$ (ratio of viscosities), $\rho_r = \dfrac{\rho_1}{\rho_2}$ (ratio of density), $\sigma_r = \dfrac{\sigma_1}{\sigma_2}$ (ratio of electrical conductivities) and $K_{T_r} = \dfrac{K_{T_1}}{K_{T_2}}$ (ratio of thermal conductivities).

The boundary conditions (5) in non-dimensional form become:

$$u_1 = 0, \qquad T_1 = 0 \qquad \text{at} \quad y = -1.$$

$$u_1 = u_2, \qquad \frac{du_1}{dy} = \frac{1}{\mu_r} \frac{du_2}{dy} \qquad \text{at} \quad y = 0.$$

$$T_1 = T_2, \qquad \frac{dT_1}{dy} = \frac{1}{K_{T_r}} \frac{dT_2}{dy} \qquad \text{at} \quad y = 0.$$

$$u_2 = 0, \qquad T_2 = 0 \qquad \text{at} \quad y = 1. \qquad \ldots (10)$$

Solution of the problem

To obtain the velocity field and temperature distribution in both the regions, we solve the coupled equations by the use of regular perturbation technique, choosing Ec as the perturbation. For the purpose, we assume (for $Ec << 1$) :

$$u_1(y) = u_{10}(y) + Ec u_{11}(y) + O(Ec)^2,$$

$$T_1(y) = T_{10}(y) + Ec T_{11}(y) + O(Ec)^2,$$

$$u_2(y) = u_{20}(y) + Ec u_{21}(y) + O(Ec)^2,$$

$$T_2(y) = T_{20}(y) + Ec T_{21}(y) + O(Ec)^2. \qquad \ldots (11)$$

Introducing (11) into the equations (6)-(9), we obtain:

Region-I ($(-1 \le y \le 0)$)

$$\frac{d^2 u_{10}}{dy^2} - M^2 u_{10} = M^2 E + P - Gr T_{10}. \qquad \ldots (12)$$

$$\frac{d^2 u_{11}}{dy^2} - M^2 u_{11} = -Gr T_{11}. \qquad \ldots (13)$$

$$\frac{d^2T_{10}}{dy^2} = 0 . \quad \dots (14)$$

$$\frac{d^2T_{11}}{dy^2} = -Pr\left(\frac{du_{10}}{dy}\right)^2 - M^2 Pr(u_{10})^2 - 2M^2 Pr\, E\, u_{10} - M^2 Pr\, E^2 . \quad \dots (15)$$

Region-II $(0 \le y \le 1)$

$$\frac{d^2u_{20}}{dy^2} - M_1^2 u_{20} = M_1^2 E + P_1 - Gr_1 T_{20}. \quad \dots (16)$$

$$\frac{d^2u_{21}}{dy^2} - M_1^2 u_{21} = Gr_1 T_{21}. \quad \dots (17)$$

$$\frac{d^2T_{20}}{dy^2} = 0. \quad \dots (18)$$

$$\frac{d^2T_{21}}{dy^2} = -M^2 Pr\frac{K_{T_r}}{\sigma_r}E^2 - M^2 Pr\frac{K_{T_r}}{\sigma_r}2E\,u_{20} - M^2 Pr\frac{K_{T_r}}{\sigma_r}u_{20}^2 - Pr\frac{K_{T_r}}{\mu_r}\left(\frac{du_{20}}{dy}\right)^2 . \quad \dots (19)$$

Introducing (11) into the boundary conditions (10), we obtain:

$$u_{10} = 0, \quad u_{11} = 0, \quad T_{10} = 1, \quad T_{11} = 0 \quad \text{at } y = -1.$$

$$u_{10} = u_{20}, \quad u_{11} = u_{21}, \quad \frac{du_{10}}{dy} = \frac{1}{\mu_r}\frac{du_{20}}{dy}, \quad \frac{du_{11}}{dy} = \frac{1}{\mu_r}\frac{du_{21}}{dy} \quad \text{at } y = 0.$$

$$T_{10} = T_{20}, \quad T_{11} = T_{21}, \quad \frac{dT_{10}}{dy} = \frac{1}{\mu_r}\frac{du_{20}}{dy}, \quad \frac{du_{11}}{dy} = \frac{1}{\mu_r}\frac{du_{21}}{dy} \quad \text{at } y = 0.$$

$$u_{20} = 0, \quad u_{21} = 0, \quad T_{20} = 0, \quad T_{21} = 0 \quad \text{at } y = 1. \quad \dots (20)$$

The solutions of coupled equations (12)-(19) under the boundary conditions (2) are obtained as follows:

$$T_{10}(y) = C_1 y + C_2 . \quad \dots (21)$$

$$T_{20}(y) = C_3 y + C_4 . \quad \dots (22)$$

$$u_{10}(y) = C_5 e^{-my} + C_6 e^{my} + K_3 y + K_4 . \quad \dots (23)$$

$$u_{20}(y) = C_7 y + C_8 + K_5 y + K_6 . \quad \dots (24)$$

$$T_{11}(y) = C_9 + C_{10}y + K_{11}e^{-2My} + K_{12}e^{2My} + K_{13}e^{-My} + K_{14}e^{My} + K_{15}ye^{My} + K_{16}ye^{-My} + K_{17}y^2 + K_{18}y^3 + K_{19}y^4. \quad \dots (25)$$

$$T_{21}(y) = C_{11} + C_{12}y + K_{20}e^{2M_1y} + K_{21}e^{-2M_1y} + K_{22}e^{M_1y} + K_{23}e^{-M_1y} + K_{24}e^{M_1y}y$$

$$+K_{25}ye^{-M_1 y}+K_{26}y^2+K_{27}y^3+K_{28}y^4.. \quad \ldots (26)$$

$$u_{11}(y)=C_{13}e^{My}+C_{14}e^{-My}+K_{33}+K_{34}e^{2My}+K_{35}e^{-2My}$$
$$+K_{36}ye^{My}+K_{37}ye^{-My}+K_{38}y^2e^{My}+K_{39}y^2e^{-My}$$
$$+K_{40}y+K_{41}y^2+K_{42}y^3+K_{43}y^4. \quad \ldots (27)$$

$$u_{21}(y)=C_{15}e^{M_1 y}+C_{16}e^{-M_1 y}+K_{44}+K_{45}e^{2M_1 y}+K_{46}e^{-2M_1 y}$$
$$+K_{47}ye^{M_1 y}+K_{48}ye^{-M_1 y}+K_{49}y^2e^{M_1 y}+K_{50}y^2e^{-M_1 y}$$
$$+K_{51}y+K_{52}y^2+K_{53}y^3+K_{54}y^4. \quad \ldots (28)$$

RESULTS AND DISCUSSION

The problem of mixed convection hydromagnetic flow of two immiscible liquids in a vertical channel with Joule heating and viscous dissipation is accomplished out in the preceding sections. The velocity and temperature fields in both the regions are obtained and expressed in (17)-(20). The equations describing the flow are governed by magnetic parameter (*M*), Grashof number (*Gr*). This enables us to carry out the numerical computations for the velocity and temperature fields for various values of the flow conditions and fluid properties.

Fig. 2 is intended to illustrate variations in the velocity versus distance *y* in region-I and region-II for different values of magnetic parameter (*m*) and electric load parameter (*E*) at fixed values ($\mu_r = 0.8$, $\rho_r = 0.6$, $\sigma_r = 0.4$, $K_{Tr} = 0.8$, $Gr = 4.0$, Pr = 1.0, P = 1.0, Ec = 0.2). It is noted that as *M* is increased the velocity is decreased in region-I as well as region-II. Also, if negative value of *E* is decreased the velocity is increased in both region but if *E* = 0.5 then flow is reverse.

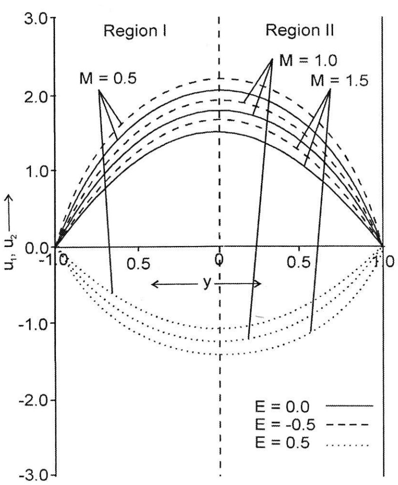

Fig. 2. Effect of electric load parameter (*E*) and magnetic parameter (*M*) on velocity field in region-I and region-II (μ_r = 0.8, ρ_r = 0.6, σ_r = 0.4, K_{Tr} = 0.8, *Gr* = 4.0. *Pr* = 1.0, *P* = 1.0 and *Ec* = 0.2).

Fig. 3 is intended to illustrate variations in the velocity versus distance y in region-I and region-II for different values of viscosity ratio (μ_r) and electric load parameter (E) at fixed values ($M = 0.5$, $\rho_r = 0.6$, $\sigma_r = 0.4$, $K_{Tr} = 0.8$, $Gr = 4.0$, $Pr = 1.0$, $P = 1.0$, $Ec = 0.2$). It is noted that as μ_r is increased the velocity is decreased in region-I as well as region-II. If electric load parameter (E) in range $(0.0 \le E \le -0.5)$ but if $E = 0.5$ then flow is reverse.

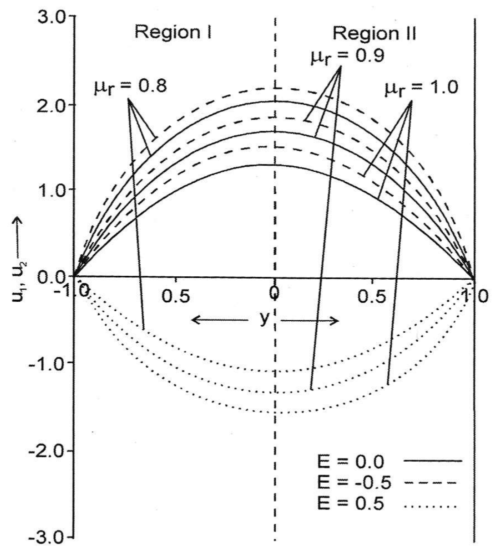

Fig. 3. Effect of electric load parameter (E) and viscosity ratio (μ_r) on velocity field in region-I and region-II (M = 0.5, ρ_r = 0.6, σ_r = 0.4, K_{Tr} = 0.8, Gr = 4.0. Pr = 1.0, P = 1.0 and Ec = 0.2).

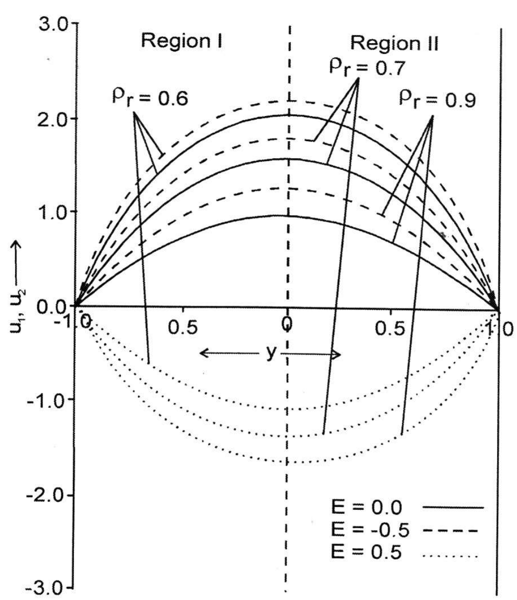

Fig. 4. Effect of electric load parameter (E) and density ratio (ρ_r) on velocity field in region-I and region-II (M = 0.5, μ_r = 0.8, σ_r = 0.4, K_{Tr} = 0.8, Gr = 4.0. Pr = 1.0, P = 1.0 and Ec = 0.2).

Fig. 4 is intended to illustrate variations in the velocity versus distance y in region-I and region-II for different values of density ratio (ρ_r) and electric load parameter (E) at fixed values ($M = 0.5$, $\mu_r = 0.8$, $\sigma_r = 0.4$, $K_{Tr} = 0.8$, $Gr = 4.0$, $Pr = 1.0$, $P = 1.0$, $Ec = 0.2$). It is noted that as ρ_r is increased the velocity is decreased in region-I as well as region-II. Also, if negative value of E is decreased the velocity is increased in both region but if $E = 0.5$ then flow is reverse.

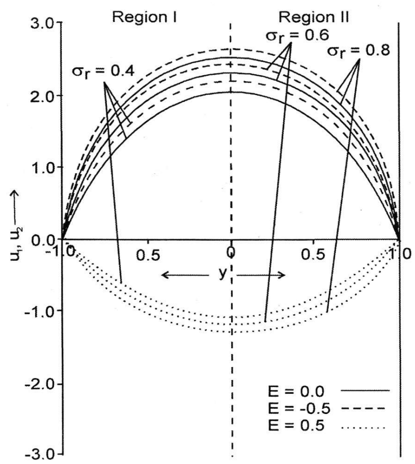

Fig. 5. Effect of electric load parameter (*E*) and electrical conductivity ratio (σ_r) on velocity field in region-I and region-II ($M = 0.5$, $\mu_r = 0.8$, $\rho_r = 0.6$, $K_{Tr} = 0.8$, $Gr = 4.0$. $Pr = 1.0$, $P = 1.0$ and $Ec = 0.2$).

Fig. 5 is intended to illustrate variations in the velocity versus distance y in region-I and region-II for different values of electrical conductivity ratio (σ_r) and electric load parameter (E) at fixed values ($M = 0.5$, $\mu_r = 0.8$, $\rho_r = 0.6$, $K_{Tr} = 0.8$, $Gr = 4.0$, $Pr = 1.0$, $P = 1.0$, $Ec = 0.2$). It is noted that as σ_r is increased the velocity is increased in region-I as well as region-II. Also, if negative value of E is decreased the velocity is increased in both region but if $E = 0.5$ then flow is reverse.

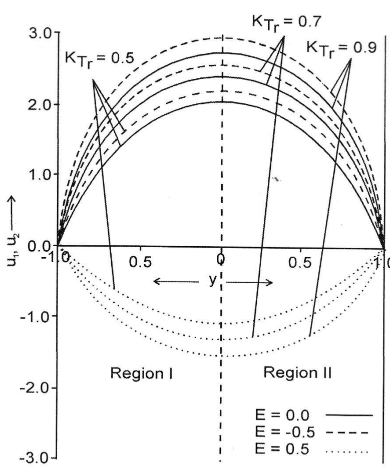

Fig. 6. Effect of electric load parameter (*E*) and thermal conductivity ratio (K_{Tr}) on velocity field in region-I and region-II ($M = 0.5$, $\mu_r = 0.8$, $\rho_r = 0.6$, $\sigma_r = 0.4$, $Gr = 4.0$. $Pr = 1.0$, $P = 1.0$ and $Ec = 0.2$).

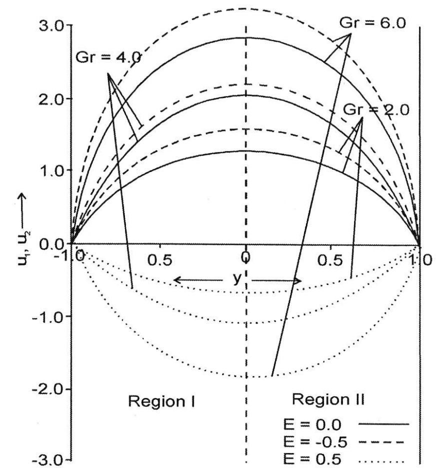

Fig. 7. Effect of electric load parameter (*E*) and Grashof number (*Gr*) on velocity field in region-I and region-II ($M = 0.5$, $\mu_r = 0.8$, $\rho_r = 0.6$, $\sigma_r = 0.4$, $K_{Tr} = 0.8$. $Pr = 1.0$, $P = 1.0$ and $Ec = 0.2$).

Fig. 6 is intended to illustrate variations in the velocity versus distance y in region-I and region-II for different values of thermal conductivity ratio (K_{Tr}) and electric load parameter (E) at fixed values ($M = 0.5$, $\mu_r = 0.8$, $\rho_r = 0.6$, $\sigma_r = 0.4$, $Gr = 4.0$, $Pr = 1.0$, $P = 1.0$, $Ec = 0.2$). It is noted that as K_{Tr} is increased the velocity is increased in region-I as well as region-II. Also, if negative value of E is decreased the velocity is increased in both region but if $E = 0.5$ then flow is reverse.

Fig. 7 is intended to illustrate variations in the velocity versus distance y in region-I and region-II for different values of Grashof number (Gr) and electric load parameter (E) at fixed values ($M = 0.5$, $\mu_r = 0.8$, $\rho_r = 0.6$, $\sigma_r = 0.4$, $K_{Tr} = 0.8$, $Pr = 1.0$, $P = 1.0$, $Ec = 0.2$). It is noted that as Gr is increased the velocity is increased in region-I as well as region-II. Also, if negative value of E is decreased the velocity is increased in both region but if $E = 0.5$ then flow is reverse.

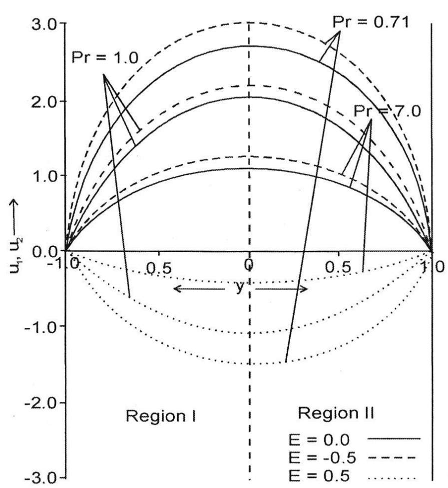

Fig. 8. Effect of electric load parameter (*E*) and Prandtl number (*Pr*) on velocity field in region-I and region-II ($M = 0.5$, $\mu_r = 0.8$, $\rho_r = 0.6$, $\sigma_r = 0.4$, $K_{Tr} = 0.8$. $Gr = 4.0$, $P = 1.0$ and $Ec = 0.2$).

Fig. 8 is intended to illustrate variations in the velocity versus distance y in region-I and region-II for different values of Prandtl number (Pr) and electric load parameter (E) at fixed values ($M = 0.5$, $\mu_r = 0.8$, $\rho_r = 0.6$, $\sigma_r = 0.4$, $K_{Tr} = 0.8$, $Gr = 4.0$, $P = 1.0$, $Ec = 0.2$). It is noted that as Pr is increased the velocity is decreased in region-I as well as region-II. Also, if negative value of E is decreased the velocity is increased in both region but if $E = 0.5$ then flow is reverse.

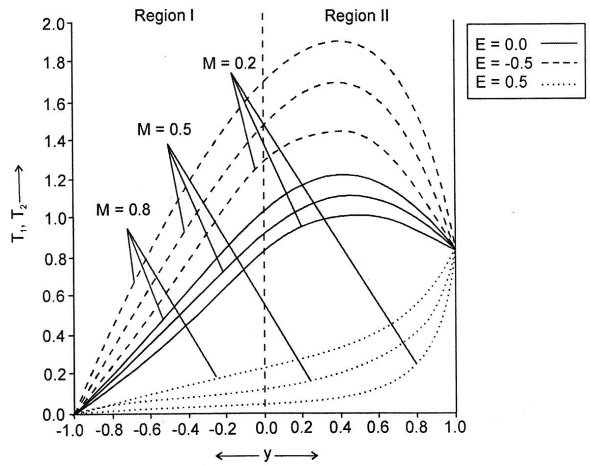

Fig. 9. Effect of electric load parameter (*E*) and magnetic parameter (*M*) on temperature profil in region-I and region-II ($\mu_r = 0.8$, $\rho_r = 0.6$, $\sigma_r = 0.4$, $K_{Tr} = 0.8$. $Gr = 4.0$, $Pr = 1.0$, $P = 1.0$ and $Ec = 0.2$).

Fig. 9 is intended to illustrate variations in the velocity versus distance y in region-I and region-II for different values of magnetic parameter (m) and electric load parameter (E) at fixed values ($\mu_r = 0.8$, $\rho_r = 0.6$, $\sigma_r = 0.4$, $K_T = 0.8$, $Gr = 4.0$, $Pr = 1.0$, $P = 1.0$, $Ec = 0.2$). It is noted that as M is increased the velocity is decreased in region-I as well as region-II. Also, if negative value of E is decreased the velocity is increased in both region but if $E = 0.5$ then flow is reverse.

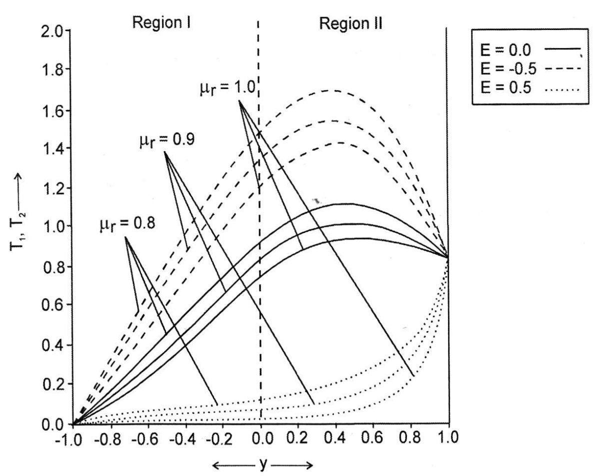

Fig. 10. Effect of electric load parameter (E) and vicosity ratio (μ_r) on temperature profil in region-I and region-II ($M = 0.5$, $\rho_r = 0.6$, $\sigma_r = 0.4$, $K_{Tr} = 0.8$, $Gr = 4.0$, $Pr = 1.0$, $P = 1.0$ and $Ec = 0.2$).

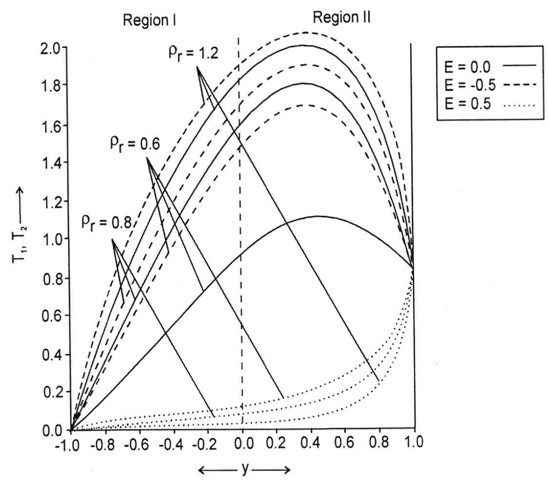

Fig. 11. Effect of electric load parameter (E) and density ratio (ρ_r) on temperature profile in region-I and region-II (M = 0.5, μ_r = 0.8, σ_r = 0.4, K_{Tr} = 0.8, Gr = 4.0, Pr = 1.0, P = 1.0 and Ec = 0.2)

Fig. 10 is intended to illustrate variations in the velocity versus distance y in region-I and region-II for different values of magnetic parameter (m) and electric load parameter (E) at fixed values ($\mu_r = 0.8,$, $\rho_r = 0.6,$ $\sigma_r = 0.4,$ $K_T = 0.8,$ $Gr = 4.0,$ $Pr = 1.0,$ $P = 1.0,$ $Ec = 0.2$). It is noted that as M is increased the velocity is decreased in region-I as well as region-II. Also, if negative value of E is decreased the velocity is increased in both region but if $E = 0.5$ then flow is reverse.

Fig. 11 is intended to illustrate variations in the velocity versus distance y in region-I and region-II for different values of magnetic parameter (m) and electric load parameter (E) at fixed values ($\mu_r = 0.8,$ $\rho_r = 0.6,$ $\sigma_r = 0.4,$ $K_T = 0.8,$ $Gr = 4.0,$ $Pr = 1.0,$ $P = 1.0,$ $Ec = 0.2$). It is noted that as M is increased the velocity is decreased in region-I as well as region-II. Also, if negative value of E is decreased the velocity is increased in both region but if $E = 0.5$ then flow is reverse.

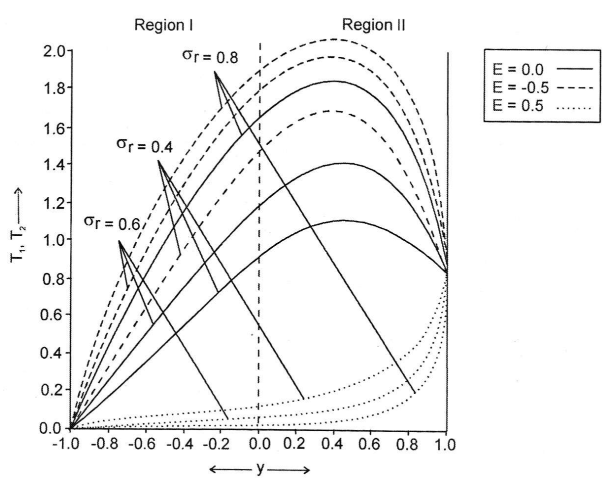

Fig. 12. Effect of electric load parameter (*E*) and electrical conductivity ratio (σ_r) on temperature profile in region-I and region-II (*M* = 0.5, μ_r = 0.8, ρ_r = 0.6, K_{Tr} = 0.8, *Gr* = 4.0, *Pr* = 1.0, *P* = 1.0 and *Ec* = 0.2).

Fig. 12 is intended to illustrate variations in the velocity versus distance *y* in region-I and region-II for different values of magnetic parameter (*m*) and electric load parameter (*E*) at fixed values (($\mu_r = 0.8$, $\rho_r = 0.6$, $\sigma_r = 0.4$, $K_T = 0.8$, *Gr* = 4.0, *Pr* = 1.0, *P* = 1.0, *Ec* = 0.2). It is noted that as *M* is increased the velocity is decreased in region-I as well as region-II. Also, if negative value of *E* is decreased the velocity is increased in both region but if $E = 0.5$ then flow is reverse.

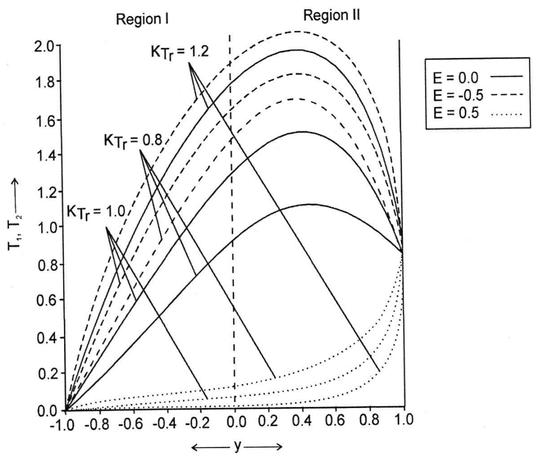

Fig. 13. Effect of electric load parameter (*E*) and thermal conductivity ratio (K_{Tr}) on temperature profile in region-I and region-II (*M* = 0.5, μ_r = 0.8, ρ_r= 0.6, σ_r = 0.4, *Gr* = 4.0, *Pr* = 1.0, *P* = 1.0 and *Ec* = 0.2).

Fig. 13 is intended to illustrate variations in the velocity versus distance y in region-I and region-II for different values of magnetic parameter (*m*) and electric load parameter (*E*) at fixed values ($\mu_r = 0.8$, $\rho_r = 0.6$, $\sigma_r = 0.4$, $K_T = 0.8$, *Gr* = 4.0, *Pr* = 1.0, *Ec* = 0.2). It is noted that as *M* is increased the velocity is decreased in region-I as well as region-II. Also, if negative value of E is decreased the velocity is increased in both region but if $E = 0.5$ then flow is reverse.

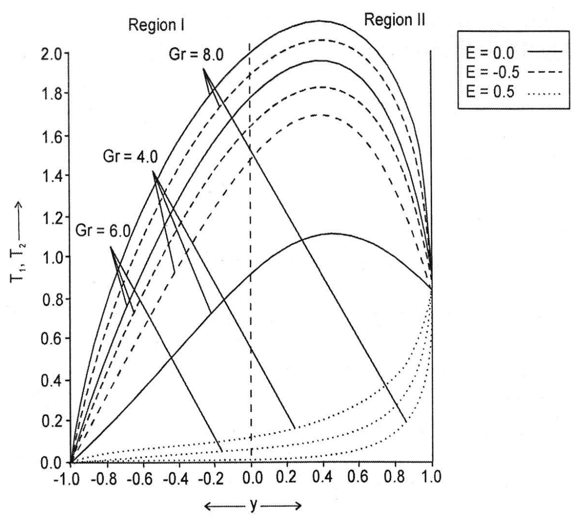

Fig. 14. Effect of electric load parameter (E) and Grashof number (Gr) on temperature profile in region-I and region-II (M = 0.5, μ_r = 0.8, ρ_r = 0.6, σ_r = 0.4, K_{Tr} = 0.8, Pr = 1.0, P = 1.0 and Ec = 0.2).

Fig. 14 is intended to illustrate variations in the velocity versus distance y in region-I and region-II for different values of magnetic parameter (m) and electric load parameter (E) at fixed values ($\mu_r = 0.8,\ \rho_r = 0.6,\ \sigma_r = 0.4,\ K_T = 0.8,\ Gr = 4.0,\ Pr = 1.0,\ P = 1.0,\ Ec = 0.2$). It is noted that as M is increased the velocity is decreased in region-I as well as region-II. Also, if negative value of E is decreased the velocity is increased in both region but if $E = 0.5$ then flow is reverse.

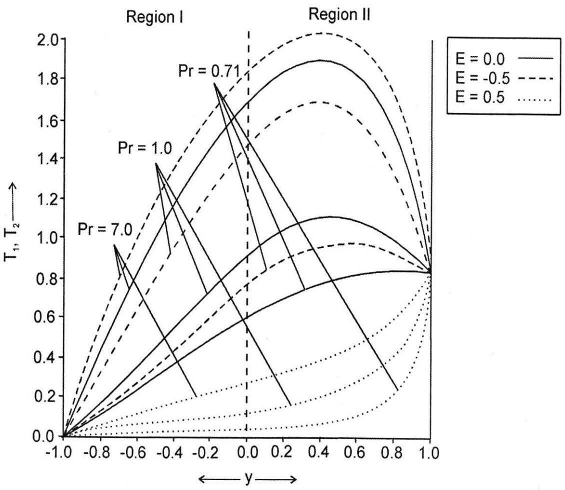

Fig. 15. Effect of electric load parameter (E) and Prandtl number (Pr) on temperature profile in region-I and region-II (M = 0.5, μ_r = 0.8, ρ_r = 0.6, σ_r = 0.4, K_{Tr} = 0.8, Gr = 4.0, P = 1.0 and Ec = 0.2).

Fig. 15 is intended to illustrate variations in the velocity versus distance y in region-I and region-II for different values of magnetic parameter (m) and electric load parameter (E) at fixed values ($\mu_r = 0.8,,\ \rho_r = 0.6,\ \sigma_r = 0.4,\ K_T = 0.8,\ Gr = 4.0,\ Pr = 1.0,\ P = 1.0,\ Ec = 0.2$). It is noted that as M is increased the velocity is decreased in region-I as well as region-II. Also, if negative value of E is decreased the velocity is increased in both region but if $E = 0.5$ then flow is reverse.

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APPENDIX

$$C_1 = -\frac{1}{1+K_{T_r}}, \qquad C_2 = \frac{K_{T_r}}{1+K_{T_r}}, \qquad C_3 = -\frac{K_{T_r}}{1+K_{T_r}}, \qquad C_4 = \frac{K_{T_r}}{1+K_{T_r}},$$

$$C_5 = \frac{\left[K_9 e^{-M} - K_8\left(K_3 - K_4\right)\right]}{K_8 e^{M} - K_7 e^{-M}}, \qquad C_6 = \frac{K_9 e^{M} - K_7\left(K_3 - K_4\right)}{K_8 e^{M} - K_7 e^{-M}},$$

$$C_7 = \frac{K_5 + K_6 + K_{10} e^{M_1}}{e^{M_1} - e^{-M_1}}, \qquad C_8 = \frac{K_5 + K_6 + K_{10} e^{-M_1}}{e^{-M_1} - e^{-M_1}},$$

$$C_9 = \frac{-K_{T_r}(K_{29} - K_{31}) + K_{30} + K_{32}}{1 + K_{T_r}} + K_{31},$$

$$C_{10} = \frac{K_{29} + K_{30} - K_{31} + K_{32}}{1 + K_{T_r}}, \quad C_{11} = \frac{-K_{T_r}(K_{29} - K_{31}) + K_{30} + K_{32}}{1 + K_{T_r}},$$

$$C_{12} = K_{30} + \frac{\left[K_{T_r}(K_{29} - K_{31}) - K_{30} - K_{32}\right]}{1 + K_{T_r}}, \; C_{13} = \frac{K_{61}e^{M} - K_{55}K_{60}}{K_{59}e^{M} - K_{60}e^{-M}},$$

$$C_{14} = \frac{K_{61}e^{-M} - K_{55}K_{60}}{K_{60}e^{-M} - K_{59}e^{M}},$$

$$C_{15} = \frac{K_{56} - K_{57}e^{-M_1}}{e^{M_1} - e^{-M_1}} - \frac{e^{-M_1}\left[\left(e^{M} - e^{-M}\right)K_{61} + (K_{59} - K_{60})K_{55}\right]}{\left(e^{M_1} - e^{-M_1}\right)\left(K_{59}e^{M} - K_{60}e^{-M}\right)},$$

$$C_{16} = \frac{K_{56} - K_{57}e^{-M_1}}{e^{-M_1} - e^{M_1}} - \frac{e^{M_1}\left[\left(e^{M} - e^{-M}\right)K_{61} + (K_{59} - K_{60})K_{55}\right]}{\left(e^{-M_1} - e^{M_1}\right)\left(K_{59}e^{M} - K_{60}e^{-M}\right)},$$

$$K_1 = -\frac{1}{1 + K_{T_r}}, \quad K_2 = -\frac{K_{T_r}}{1 + K_{T_r}}, \quad K_3 = \frac{K_1 Gr}{M^2}, \quad K_4 = -\frac{\left(M^2 E + P - Gr K_2\right)}{M^2},$$

$$K_5 = \frac{K_2 Gr_1}{M_1^2}, \; K_6 = -\frac{\left(M_1^2 E + P - K_2 Gr_1\right)}{M_1^2}, \; K_7 = M\left[4e^{M}(\mu_r + 1)(e^{-M} - e^{M})\right],$$

$$K_8 = M\left[2\left(e^{M} + e^{-M}\right) - \left(e^{M} - e^{-M}\right)(\mu_r - 1)\right],$$

$$K_9 = 2M\Big[K_3 - \left(1 + e^{M}\right)K_4 - K_5 + \left(e^{M} - 1\right)K_6$$
$$- \left(e^{-M} - e^{M}\right)\left(K_5 - \mu_r K_3 - MK_6 + MK_4\right)\Big],$$

$$K_{10} = -K_6 + K_4 - \left[\frac{\left(e^{M} - e^{-M}\right)K_9 - (K_7 + K_8)(K_3 + K_4)}{K_8 e^{M} - K_7 e^{-M}}\right], \quad K_{11} = -\frac{Pr\, C_5}{2},$$

$$K_{12} = -\frac{Pr\, C_6}{2}, \; K_{13} = \left[-\frac{2Pr\, C_5 K_3}{M} - 2EPr\, C_5 - 2C_5 K_4 Pr\right],$$

$$K_{14} = \left[\frac{2Pr\, C_6 K_3}{M} - 2EPr\, C_6 - 2C_6 K_4 Pr\right], \quad K_{15} = \left[-2Pr\, C_5 K_3 M^2\left(1 - \frac{2}{M}\right)\right],$$

$$K_{16} = \left[2Pr\, C_5 K_3 M^2\left(1 + \frac{2}{M}\right)\right],$$

$$K_{17} = -\frac{Pr\left(K_3^2 + 2M^2 C_5 C_6\right) - M^2 Pr E^2}{2} - EM^2 PrK + C_5 C_6 M^2 Pr + K_4 M^2 Pr,$$

$$K_{18}=\frac{K_3K_4M^2Pr-EM^2PrK_3}{3},\quad K_{19}=-\frac{K_3^2M^2Pr}{12},$$

$$K_{20}=-M^2Pr\frac{K_{T_r}}{\sigma_r}\frac{C_8^2}{4M_1^2}-Pr\frac{K_{T_r}}{\mu_r}\frac{C_8^2}{4},\quad K_{21}=-M^2Pr\frac{K_{T_r}}{\sigma_r}\frac{C_7^2}{4M_1^2}-Pr\frac{K_{T_r}}{\mu_r}\frac{C_7^2}{4},$$

$$K_{22}=2M^2Pr\frac{K_{T_r}}{\sigma_rM_1}\left[-\frac{C_8}{M_1}+\frac{C_8K_6}{M_1}+C_8K_5\right],$$

$$K_{23}=-M^2Pr\frac{K_{T_r}}{\sigma_rM_1}\left[-\frac{C_7}{M_1}+\frac{2C_7K_6}{M_1}+2C_7K_5\right],$$

$$K_{24}=-2M^2Pr\frac{K_{T_r}}{\sigma_r}C_8K_5\left(1-\frac{2}{M_1}\right),\quad K_{25}=-2M^2Pr\frac{K_{T_r}}{\sigma_r}C_7K_5\left(1+\frac{2}{M_1}\right),$$

$$K_{26}=-M^2Pr\frac{K_{T_r}}{\sigma_r}\left[\frac{E^2}{2}+K_6-\frac{K_6^2}{2}-C_7C_8-\frac{K_5^2}{2}+M_1^2C_7C_8\right],$$

$$K_{27}=-M^2Pr\frac{K_{T_r}}{\sigma_r}\left[\frac{K_5+K_5K_6}{3}\right],\quad K_{28}=-\frac{M^2PrK_{T_r}K_5^2}{12\sigma_r},$$

$$K_{29}=-K_{11}e^{2M}-K_{12}e^{-2M}-(K_{13}-K_{16})e^{M}-(K_{14}-K_{15})e^{-M}-K_{17}+K_{18}-K_{19},$$

$$K_{30}=-\Big[K_{20}e^{2M_1}+K_{21}e^{-2M_1}+(K_{22}+K_{24})e^{M_1}$$
$$+(K_{23}+K_{25})e^{-M_1}+K_{26}+K_{27}+K_{28},$$

$$K_{31}=K_{20}+K_{21}+K_{22}+K_{23}+K_{24}-K_{11}-K_{12}-K_{13}-K_{14},$$

$$K_{32}=C_{12}+M_1(2K_{20}-2K_{21}+K_{22}-K_{23})+K_{24}+K_{25}$$
$$+MK_{T_r}(2K_{11}-2K_{12}+K_{13}-K_{14})-K_{15}-K_{16},$$

$$K_{33}=\frac{Gr}{M}\left[C_9+\frac{2K_{17}}{M^2}+\frac{24K_{19}}{M^4}\right],\quad K_{34}=-\frac{GrK_{12}}{3M^2},\quad K_{35}=-\frac{GrK_{11}}{3M^2},$$

$$K_{36}=\frac{Gr}{2M}\left[K_{14}+\frac{K_{15}}{2M}\right],\quad K_{37}=\frac{Gr}{2M}\left[K_{13}+\frac{K_{16}}{2M}\right],\quad K_{38}=-\frac{GrK_{15}}{4M},$$

$$K_{39}=-\frac{GrK_{16}}{4M},\ K_{40}=\frac{Gr}{M^2}\left[C_{10}+\frac{6K_{18}}{M^2}\right],\ K_{41}=\frac{Gr}{M^2}\left[K_{17}+\frac{12K_{19}}{M^2}\right],$$

$$K_{42}=\frac{GrK_{18}}{M^2},\ K_{43}=\frac{GrK_{19}}{M^2},\ K_{44}=\frac{Gr_1}{M_1^2}\left[C_{11}+\frac{2K_{26}}{M_1^2}+\frac{24K_{28}}{M_1^4}\right],$$

$$K_{45}=-\frac{Gr_1K_{20}}{3M_1^2},\quad K_{46}=-\frac{Gr_1K_{21}}{3M_1^2},\quad K_{47}=\frac{Gr_1}{2M_1}\left[K_{22}+\frac{K_{24}}{2M_1}\right],$$

$$K_{48} = \frac{Gr_1}{2M_1}\left[K_{23} + \frac{K_{25}}{2M_1}\right], \quad K_{49} = -\frac{Gr_1 K_{24}}{4M_1}, \quad K_{50} = \frac{Gr_1 K_{25}}{4M_1},$$

$$K_{51} = \frac{Gr_1}{M_1^2}\left[C_{12} + \frac{6K_{27}}{M_1^2}\right], \quad K_{52} = \frac{Gr_1}{M_1^2}\left[K_{26} + \frac{12K_{28}}{M_1^2}\right], \quad K_{53} = \frac{Gr_1 K_{27}}{M_1^2},$$

$$K_{54} = \frac{Gr_1 K_{28}}{M_1^2}, \qquad K_{55} = -\Big[K_{33} + K_{34}e^{-2M} + K_{35}e^{2M} - (K_{36} - K_{38})e^{-M}$$

$$-(K_{37} - K_{39})e^{M} + K_{40} + K_{41} - K_{42} + K_{43},$$

$$K_{56} = -\Big[K_{44} + K_{45}e^{2M_1} + K_{46}e^{-2M_1} + (K_{47} + K_{49})e^{M_1}$$

$$+(K_{48} + K_{50})e^{-M_1} + K_{51} + K_{52} + K_{53} + K_{54},$$

$$K_{57} = K_{33} + K_{34} + K_{35} - K_{44} - K_{45} - K_{46},$$

$$K_{58} = 2M_1K_{45} - 2M_1K_{46} + K_{47} + K_{48} + K_{51}$$

$$-\mu_r(2MK_{34} - 2MK_{35} + K_{36} + K_{37} + K_{40}),$$

$$K_{59} = 2M_1(e^{M_1} - e^{-M}) - (\mu_r + M_1)(e^{-M_1} - e^{M_1}),$$

$$K_{60} = 2M(e^{M} - e^{M_1}) + (\mu_r + M_1)(e^{-M_1} - e^{M_1})$$

$$K_{61} = 2M(K_{55} + K_{56} - K_{57}e^{M_1}) - (e^{-M_1} - e^{M_1})(M_1K_{57} + K_{56}).$$

❑