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"USING DIOPHANTINE EQUATION ANALYSIS OF UNIQUE FACTORIZATION IN THE RING OF INTEGERS OF CERTAIN QUADRATIC FIELDS"

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It is provided that Diophantine equation $x^2 + 11 = 3^n$ and $x^2 + 19 = 7^n$ has the solution for n = 3 and the equation $x^2 + 17 = 4^n$ have no integer solution. In this paper we analysis these unique factorization in Diophantine equation by Nagell's idea method.

KEYWORDS: Diophantine equation, Nagell's idea, Divisors, factors etc.

INTRODUCTION

In this paper we are considered the equation $x^2 + 7 = 2^n$ has solution only for $n = 3, 4, 5, 7$ and 15. An elementary proof, based on Negell's idea is given for the following results and other equation be $x^2 + 29 = 13^n$ the non existence of solution for this equation is provided using unique factorization in the ring of integers θd of $\theta\left(\sqrt{d}\right)$ for $d = 29$ the method of proof is similar to the one we have proved in other

THEOREM

The only integer solutions of the equation

$$x^2 + 19 = 7^n \qquad \ldots (1)$$

and

$$x^2 + 11 = 3^n \qquad \ldots (2)$$

are $(x, n) = (\pm 18, 3)$ and $(\pm 4, 3)$ respectively.

Proof : We first deal with equation (1) if (x, n) is a solution of equation (1) then x is even. Hence $x^2 + 19 \equiv 3 \ (Mod\, 4)$ which implies that n is odd when n = 1 given equation has no solution in integers

We rewrite the equation as $x^2 + 19 = 7^n$

$$\left(x + \sqrt{-19}\right)\left(x - \sqrt{-19}\right) = 7^n \qquad \ldots (3)$$

It is an equation is $\theta\left(\sqrt{-19}\right)$ where ring of integers θ_{-19} has unique factorization.

Let $\alpha = \frac{3+\sqrt{-19}}{2} \qquad \beta = \frac{3-\sqrt{-19}}{2}$

α, β satisfy the equation $\alpha+\beta = 3, \ \alpha\beta = 7$

$$\alpha^2 - 3\alpha + 7 = 0 \quad \text{and} \quad \beta^2 - 3\beta + 7 = 0$$

by (3)
$$\left(x+\sqrt{-19}\right)\left(x-\sqrt{-19}\right) = \alpha^n\beta^n \qquad \ldots (4)$$

Any common divisor

z of $\left(x-\sqrt{-19}\right)$ and $\left(x+\sqrt{-19}\right)$ divides their difference which $\Rightarrow N(z)$ divides $N\left(2\sqrt{-19}\right) = 76.$ As $N(\alpha) = N(\beta) = 7$

$$\left(x+\sqrt{-19}\right) = \pm\alpha^n \quad \text{or} \quad \beta^n \text{ and } \left(x-\sqrt{-19}\right) = \pm\,\beta^n \text{ or } \alpha^n$$

$$\alpha^n - \beta^n = \pm 2(\alpha-\beta) \qquad \ldots (5)$$

The equation $\alpha^2 = 3\alpha - 7, \beta^2 = 3\beta - 7$ the power of α and β are given by the formulas

$$\alpha^n = r_n\alpha + \delta_n \quad \beta^n = r_n\beta + s_n \qquad \forall n \geq 3 \qquad \ldots (6)$$

where
$$\begin{pmatrix} r_{n+1} \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -7 & 0 \end{pmatrix}\begin{pmatrix} r_n \\ s_n \end{pmatrix} \ \forall \ n > 3$$

and
$$\begin{pmatrix} r_n \\ s_n \end{pmatrix} = \begin{pmatrix} 2 \\ -21 \end{pmatrix}$$

thus, we have
$$\begin{pmatrix} r_n \\ s_n \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -7 & 0 \end{pmatrix}^{n-3}\begin{pmatrix} r_3 \\ s_3 \end{pmatrix} \qquad \ldots (7)$$

by (6)
$$\Rightarrow \alpha^n - \beta^n = r_n(\alpha-\beta) \qquad \ldots (8)$$

Using (5) we get
$$r_n = \pm 2 \qquad \ldots (9)$$

if
$$A = \begin{pmatrix} 3 & 1 \\ -7 & 0 \end{pmatrix} \text{ then } A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \bmod 2$$

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \bmod 2$$

$$A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2$$

As $n > 3$ and n is odd A^{n-3} take only value $A^2 \bmod 2$ and by (7) we get

$$\begin{pmatrix} r_n \\ S_n \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \bmod 2$$

$$\forall n > 3 \quad r_n \equiv 1 \bmod 2 \quad \text{so that } r_n \neq \pm 2$$

equation (1) has no solution for $n > 3$.

pr $n = 2$ equation (2) has no solution and $n = 3$, $x = \pm 4$ satisfy (2) then $\alpha = \frac{1+\sqrt{-11}}{2}$, $\beta = \frac{1-\sqrt{-11}}{2}$ we proof as above.

Theorem II. The equation $x^2 + 43 = 11^n$ has no solution of integers

Proof : If *(x, n)* is a solution of (1,1) then *x* is even. Hence, $x^2 + 43 = 3(\text{mod } 4)$ which $\Rightarrow$ that *n* is odd when *n* = 1, 3 the given equation has no solution in integers. Now assume that $n \geq 5$ we rewrite the equation

$$\left(x+\sqrt{-43}\right)\left(x-\sqrt{-43}\right) = 11^n \qquad \ldots (1)$$

is an equation in $\theta\left(\sqrt{-23}\right)$ whose ring of integers θ_{23} has unique factorization $\alpha + \beta = 1 \quad \alpha\beta = 11$

Let $\alpha = \frac{1+\sqrt{-43}}{2}$, $\beta = \frac{1-\sqrt{-43}}{2}$ $\alpha_1\beta$ are satisfy the equation

$$\alpha^2 - \alpha + 11 = 0 \text{ and } \beta^2 - \beta + 11 = 0 \qquad \ldots (2)$$

Then equation $\left(x+\sqrt{-43}\right)\left(x-\sqrt{-43}\right) = \alpha^n\beta^n$ … (3)

As common divisor *z* of $\left(x+\sqrt{-43}\right)$ and $\left(x-\sqrt{-43}\right)$ divides their difference $\Rightarrow N(\zeta)$ divides $N\left(\alpha\sqrt{-43}\right) = 172$ as $N(\alpha) = N(\beta) = 11$ neither α and nor β is common divisor of $\left(x-\sqrt{-43}\right)$ and $\left(x+\sqrt{-43}\right)$. A θ_{-43} is a unique factorization domain whose only units are $\pm$ 1, we have

$$\left(x+\sqrt{-43}\right) = \pm\alpha^n \text{ or } \pm\beta^n \text{ and } \left(x-\sqrt{-43}\right) = \beta^n \text{ or } \alpha^n \qquad \ldots (4)$$

can be obtain $\alpha^n - \beta^n = \pm 2(\alpha - \beta)$ … (5)

Let $\alpha^n = r_n\alpha + S_n \quad \beta^n = r_n\beta + S_n \quad \forall n \geq 3$

$$\begin{pmatrix} r_{n+1} \\ S_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -11 & 0 \end{pmatrix} \begin{pmatrix} r_n \\ S_n \end{pmatrix} . \forall n \geq 3$$

and $\begin{pmatrix} r_3 \\ S_3 \end{pmatrix} = \begin{pmatrix} -10 \\ -11 \end{pmatrix}$

Thus we have $\begin{pmatrix} r_n \\ S_n \end{pmatrix} = \begin{pmatrix} +1 & 1 \\ -11 & 0 \end{pmatrix}^{n-3} \begin{pmatrix} r_3 \\ S_3 \end{pmatrix} \forall n \geq 3$

$$\alpha^n - \beta^n = r_n(\alpha - \beta) \qquad \ldots (6)$$

by (5) and (6) $r_n = \pm 2$

if $A = \begin{pmatrix} 1 & 1 \\ -11 & 0 \end{pmatrix}$ then $A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \text{mod } 2$

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \bmod 2 \text{ and } A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2$$

as $n > 3$ n is odd A^{n-3} takes only value of A^2 mod 2

$$\begin{pmatrix} r_n \\ S_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \bmod \forall n \geq 3 \;\; r_n = 1 \bmod 2$$

So fercit $r_n \neq 2$ theorem is proof we share show that for other equation as $x^2 + 67 = 19^n$

References

1. Karlheinz, Spindler, *Abstract Algebra with Applications*, Volume **II**, Marcel Dekker inc.
2. Kasahara, S., On some generalizations of the Branch Contraction theorem, *Math Seminar Notes*, **3**, 161–169 (1975).
3. Skof, F., Terana di punti Fisso per applicazioni negli spazi matric, *Atti. Acad. Sci*., Torino, **111**, 323–329 (1977).
4. Bhatta, S. Parameshwara and Shashirekha, H., A characterization of completeness for trellises, *Algebra Universatis,* **44**, P. 305–308 (2000).

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