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"COMPLEX REPRESENTATIONS OF CIRCLE AND STRAIGHT LINES IN ARGAND PLANES"

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In this paper we are interested in the equation of circle and equation of straight line at a point of a smooth curve in the argand plane. Instead of rewriting the equation in real variables, we are solved the same example by complex number and represent straight line and circle for given point.

KEYWORDS: Circle, Straight line, Concylic, real and imaginary

INTRODUCTION

Let Z be any point on the straight line joining z_1 and z_2.

Then

(1) $\arg \frac{\overrightarrow{PA}}{\overrightarrow{PB}} = \pi$ (P is an internal point)

$$\Rightarrow \arg \frac{z - z_1}{z_2 - z} \text{ or } \arg \frac{z_1 - z}{z_2 - z} = \pi$$

$$\Rightarrow \arg \frac{z_1 - z}{z_2 - z} \text{ is purly real.}$$

(2) $\arg \frac{\overrightarrow{PA}}{\overrightarrow{PB}} = 0$ is P is an internal point

$$\Rightarrow \arg \frac{z - z_1}{z - z_2} = 0 \qquad \frac{z - z_1}{z - z_2} \text{ purly real}$$

Hence whether the part P is internal of internal in AB.

Now $\frac{z - z_1}{z - z_2} = \text{real} \Rightarrow \text{Img}\left[\frac{z - z_1}{z - z_2}\right] = 0$

$$\Rightarrow \frac{1}{2i}\left\{\frac{z - z_1}{z - z_2} - \left(\overline{\frac{z - z_1}{z - z_2}}\right)\right\} = 0$$

where $\quad \text{Img } z = \dfrac{z - \bar{z}}{2i}$

and $\quad \overline{z - z_1} = \bar{z} - \bar{z}_2$

$$\frac{z - z_1}{z - z_2} = \left(\overline{\frac{z - z_1}{z - z_2}}\right) = \left(\frac{\bar{z} - \bar{z}_1}{\bar{z} - \bar{z}_2}\right)$$

$$(z - z_1)(\bar{z} - \bar{z}_2) = (z - z_2)(\bar{z} - \bar{z}_1)$$

$$\cancel{z\bar{z}} - z\bar{z}_2 - z_1\bar{z} + z_1\bar{z}_2 = \cancel{z\bar{z}} - z\bar{z}_1 - z_2\bar{z} + \bar{z}_1 z_2$$

$$z\bar{z}_1 - z\bar{z}_2 - \bar{z}z_1 + \bar{z}z_2 + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$$

$$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0 \qquad \ldots (1)$$

is called equation of straight line in complex plane with point z_1, z_2

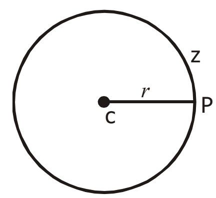
Multiplying by i of (1)

$$iz(\bar{z}_1 - \bar{z}_2) - i\bar{z}(z_1 - z_2) + iz_1\bar{z}_2 - iz_2\bar{z}_1 = 0$$

Let us assume $-i(z_1 - z_2) = q$ and $i(\bar{z}_1 - \bar{z}_2) = \bar{q}$ and $i(z_1\bar{z}_2 - z_2\bar{z}_1) = r$

$$z\bar{q} + q\bar{z} + r = 0 \quad q \neq 0 \quad r \text{ is}$$

Let $\quad z = x + iy$ and $\bar{z} = x - iy$



for any point of circle $|z - a| = CP$

$$|z - a|^2 = r^2 \quad CP = r \text{ (Radius)}$$

$$(z - a)(\overline{z - a}) = r^2 \text{ or } (z - a)(\bar{z} - \bar{a}) = r^2$$

$z\bar{z} - \bar{a}z - a\bar{z} + a\bar{a} - r^2 = 0$ is called equation of circle General equation of circle putting $c = a\bar{a} - r^2$

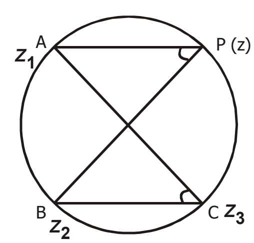
then $\quad z\bar{z} - \bar{a}z - a\bar{z} + c = 0$ put $b = -a$

$$z\bar{z} + \bar{b}z + b\bar{z} + c = 0$$

Equation of circle passing through three points *A*, *B* and *C*.

points are $z_1 = A$, $z_2 = B$ and $z_3 = C$

By fig.



$$\angle ACB = \angle APB$$

$$\arg \frac{\overrightarrow{CB}}{\overrightarrow{CA}} = \arg \frac{\overrightarrow{PB}}{\overrightarrow{PA}}$$

$$\arg \frac{z_2 - z_3}{z_1 - z_3} = \arg \frac{z_2 - z}{z_1 - z}$$

or $$\arg\frac{z_3-z_1}{z_3-z_2}=\arg\frac{z-z_1}{z-z_2}$$

$$\arg\left\{\frac{z_3-z_1}{z_3-z_2}\Big/\frac{z-z_1}{z-z_2}\right\}=0$$

or $$\left(\frac{\overline{z}_3-\overline{z}_1}{\overline{z}_3-\overline{z}_2}\right)\Big/\left(\frac{\overline{z}-\overline{z}_1}{\overline{z}-\overline{z}_2}\right)=0$$

required equation of circle passes through z_1, z_2, z_3 points z_1, z_2, z_3 are in concyclic order, then,

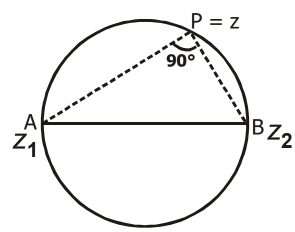
$$\text{Im}\left\{\left(\frac{z_3-z_1}{z_3-z_2}\right)\Big/\left(\frac{z-z_1}{z-z_2}\right)\right\}=0$$

$$\Rightarrow \left(\frac{z_3-z_1}{z_3-z_2}\right)\Big/\left(\frac{z-z_1}{z-z_2}\right) \text{ is purely real}$$

To find equation circle taining the point z_1 and z_2 as a diameter

Let P be any point on the circle where diameter is AB

$$A=z_1 \quad\text{and}\quad B=z_2$$



then $$\arg\left(\frac{\overrightarrow{PB}}{\overrightarrow{PA}}\right)=\frac{\pi}{2} \text{ (by property of Geometry)}$$

$$\arg\left[\frac{z_2-z}{z_1-z}\right] \text{ is purely Imaginary.}$$

then

$$\text{Rel}\left(\frac{z_2-z}{z_1-z}\right)=0$$

$$\frac{1}{2}\left\{\frac{z_2-z}{z_1-z}+\left(\overline{\frac{z_2-z}{z_1-z}}\right)\right\}=0$$

$$\frac{z_2-z}{z_1-z}=-\frac{\overline{z}_2-\overline{z}}{\overline{z}_1-\overline{z}}$$

$$2z\overline{z}-z\left(\overline{z}_1+\overline{z}_2\right)-\overline{z}\left(z_1+z_2\right)+\overline{z}_1z_2+z_2\overline{z}_1=0$$

Ex. (1). *Prove that* $\left|\frac{z-1}{z+1}\right|=$ *constant and amp.* $\left(\frac{z-1}{z+1}\right)=$ *constant are orthogonal circles.*

Ans. $\left|\frac{z-1}{z+1}\right|=\lambda$ (say) $\Rightarrow\left|\frac{x-1+iy}{x+1+iy}\right|=\lambda$

$$\frac{(x-1)^2+y^2}{(x+1)^2+y^2}=\lambda^2$$

$\Rightarrow x^2+y^2+2\left(\frac{\lambda^2+1}{\lambda^2-1}\right)x+1=0$ which is the form of $x^2+y^2+2yx+1=0$ and represents a circle.

again $\quad \text{amp}\left(\frac{z-1}{z+1}\right)=\text{constant}$

$$\Rightarrow \text{amp}\,(z-1)-\text{amp}\,(z+1)=\text{constant}$$

$$\Rightarrow \text{amp}\,(x-1+iy)-\text{amp}\,(x+1+iy)=\text{constant}$$

$$\Rightarrow \tan^{-1}\frac{y}{x-1}-\tan^{-1}\frac{y}{x+1}=\text{constant}$$

$$\Rightarrow \tan^{-1}\frac{2y}{x^2+y^2-1}=\text{constant}$$

$$\Rightarrow \frac{2y}{x^2+y^2-1}=\mu$$

$\Rightarrow x^2+y^2-\frac{2}{\mu}y-1=0$ this circle is form of $x^2+y^2+2fy+c=0$

Ex. 2. *Find all the circles which are orthogonal to* $|z|=1$ and $|z-1|=4$.

Ans. Let $|z-2|=l$ where $2=a+ib$ where *a, b* and *k* be the circle which cuts orthogonally, then using property that the sum of square of their radii is equal to the square of the distance between their centres, $|z-2|=k$

$$|z|=1 \text{ or } |z-0|=1$$

$$|\alpha-0|^2=k^2+1=\alpha\bar{\alpha}$$

and $\quad k^2+16=|\alpha-1|^2=(\alpha-1)(\bar{\alpha}-1)$

$$k^2+1=(\alpha-1)(\bar{\alpha}-1)-15$$

$$\cancel{\alpha\bar{\alpha}}=\cancel{\alpha\bar{\alpha}}-\bar{\alpha}-\alpha+1-15$$

$$\alpha+\bar{\alpha}=-14$$

$$a+\cancel{ib}+a-\cancel{ib}=-14$$

$$2a=-14 \Rightarrow a=-7$$

Now $\quad \alpha=a+ib=-7+ib$

also $\quad k^2=\alpha\bar{\alpha}-1=(-7+ib)(-7-ib)-1$

$$=49+b^2-1=48+b^2$$

required family of circles $|z+7-bi| = \sqrt{48+b^2}$

Conclusion

In this paper we shall solve the given problems of circle and straight line in complex representation using $z = x + iy$, $\overline{z} = x - iy$.

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