THE CLOSURE GRAPH ON α^{s*}-OPEN SETS

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The purpose of this paper is to make the new type of Graph by using topological space with α^{s}*-open sets and investigate some of their basic properties and give characterizations of them.

KEYWORDS AND PHRASES : α^{s}*-open, α^{s}*-closed, α^{s}*-continuous, α^{s}*-homeomorphism, α^{s}*-closure graph.

INTRODUCTION

The notion of α-open sets was introduced by O. Njastad [7] in 1965. Quite recently, the authors introduced [2] a new class of nearly open set namely α^{s}*-open sets and studied some functions using these sets. Also introduce [3] the concepts of α^{s}*-continuous, *M*-α^{s}*-continuous, α^{s}*-connected space and introduce [4] α^{s}*-regular, α^{s}*-normal space.

Graph theory may be said to have its beginning in 1736 when EULER considered the (general case of the) Königsberg bridge problem. It has developed into an extensive and popular branch of mathematics, which has been applied to many problems in mathematics, computer science, and other scientific and not-so-scientific areas.

A graph *G* is an ordered (*V* (*G*), *E* (*G*)) consisting of a nonempty set *V* (*G*) of vertices, a set *E* (*G*), disjoint from *V* (*G*), of edges and an incidence function that associates with each edge of *G* an unordered pair of (not necessarily distinct) vertices of *G*. If *e* is an edge and *u* and *v* are vertices such that (*e*) *uv*, then *e* is said to join *u* and *v*.

In this paper, we introduce α^{s}*-closure graph by using topological space with α^{s}*-open sets and investigate some of their basic properties.

PRELIMINARIES

Definition 2.1 [6]. A subset *A* of a topological space (X, τ) is called

(i) generalized closed (briefly *g*-closed) if cl $(A) \subseteq U$ whenever $A \subseteq U$ and *U* is open in *X*.

(ii) generalized open (briefly *g*-open) if $X \backslash A$ is *g*-closed in *X*.

Definition 2.2 [6]. Let *A* be a subset of *X*. The generalized closure of *A* is defined as the intersection of all *g*-closed sets containing *A* and is denoted by cl* (*A*).

Definition 2.3 [3]. A subset A of a topological space (X, τ) is called

(i) α^s* open [2] if $A \subseteq \text{int}^* (\text{cl} (\text{int} (A)))$.

(ii) α^s* closed [2]) if $X \backslash A$ is α^s* open or equivalently if $\text{cl}^* (\text{int} (\text{cl} (A))) \subseteq A$.

Definition 2.4 [2]. Let A be a subset of X. Then the α^s*-closure of A is defined as the intersection of all α^s* closed sets containing A and it is denoted by $\alpha^s* \text{cl} (A)$.

Theorem 2.5 [2]. (i) Every open set is α^{s*} open and every closed set is α^{s*} closed.

Theroem 2.6 [2]. If $\{A_\alpha\}$ is a collection of α^{s*}-open sets in X, then $\bigcup A\alpha$ is also α^{s*}-open in X.

Definition 2.7 [3]. A function $f : X \to Y$ is said to be

(i) α^{s*}-continuous if $f^{-1}(V)$ is α^{s*} open in X. For every open set V in Y.

(ii) M- α^{s*}-continuous if $f^{-1}(V)$ is α^{s*} open in X. For every α^{s*}-open set V in Y.

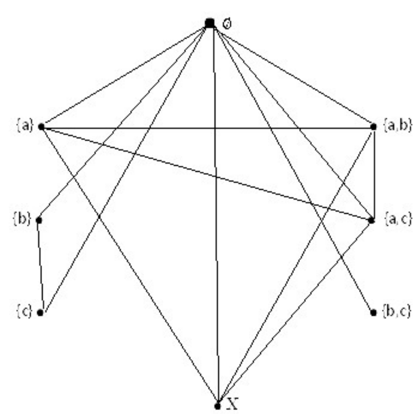
Definition 2.8 [2]. A topological space X is said to be α^s*-connected if X cannot be expressed as the union of two disjoint non-empty α^s*-open sets in X.

Theorem 2.9 [2]. A topological space X is α^s*-connected if and only if the only α^s*-regular subsets of X are $\emptyset$ and X itself.

A^{S*}-CLOSURE GRAPH

Definition 3.1. Let (X, τ) be a topological space. The α^s*-Closure Graph α^s*-$CLG\ (X, \tau)$ of a topological space (X, τ) is a graph with vertex set $P(X)$ and two distinct vertices A and B are adjacent in α^s*-$CLG\ (X, \tau)$ if $\alpha^s*\text{-Cl}(A) \cap \alpha^s*\text{-Cl}(B)$ is α^s*-open.

Example 3. 2. Let $X = \{a, b, c\}$. Let $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ then the α^s*-Closure Graph α^s*-$CLG\ (X, \tau)$ is given below.



Proposition 3.3. Let (X, τ) be a topological space. In α^s*-Closure Graph α^s*-$CLG\ (X, \tau)$, $\Delta (G) = |P(X)| - 1$.

Proof : Let A be any set. Then $\emptyset \cap A = \emptyset$ is α^s*-open which implies is of full degree. Therefore $\Delta (G) = |P(X)| - 1$.

Corollary 3.4. α^s*-closure graph α^s*-$CLG\ (X, \tau)$ is connected.

Proof : It follows from the proposition.

Definition 3.5. A subset A of X is α^s*-dense if α^s*cl $(A) = X$.

Proposition 3. If (X, τ) has a non-trivial α^s*-dense subset, then α^s*-CLG (X, τ) cannot be a tree.

Proof : Since (X, τ) has a non-trivial α^s*-dense subset namely A. clearly $\{\emptyset, X, A\}$ form a cycle in α^s* CLG (X, τ). Hence α^s*-closure graph α^s* CLG (X, τ) –closure graph cannot be a tree.

Definition 3.7. If $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ is said to be M-α^s*-homeomorphism if both f and f^{-1} are M- α^s*-continuous.

Theorem 3.8. Let X be a finite set. Let (X, τ_1) and (Y, τ_2) be topological spaces. If (X, τ_1) is M- α^s*-homeomorphic to (Y, τ_2), then α^s* CLG $(X, \tau_1) \cong \alpha^s$* CLG (Y, τ_2).

Proof : Let $f : (X, \tau_1) \rightarrow (Y, \tau_2)$ be a M-α^s*-homeomorphism. Let A and B are adjacent in α^s* CLG (X, τ_1).

Define $\tilde{f} : P(X) \rightarrow P(Y)$ $\tilde{f}(A) = f(A)$ for all $A \in P(X)$.

Claim : $\tilde{f}$ is bijection.

Let $A, B \in P(X)$ and let $\tilde{f}(A) = \tilde{f}(B)$

Let $x \in A$ $\iff f(x) \in f(A) = \{f(y)/y \in B\}$

$\iff f(x) \in \tilde{f}(A)$

$\iff f(x) \in \tilde{f}(B)$

$\iff x \in B$

Therefore $A = B$. Hence $\tilde{f}$ is one-one

Clearly $\tilde{f}$ is onto

Claim $f(\alpha^s$* cl $(A)) = \alpha^s$* cl $(f(A))$

Since f is M-α^s*-continuous, $f(\alpha^s$* cl $(A)) \subseteq \alpha^s$* cl $(f(A))$

Since f is M-α^s*-homeomorphism, $f(\alpha^s$* cl $(A))$ is α^s*-closed set.

Also $A \subseteq \alpha^s$* cl $(A) \Rightarrow f(A) \subseteq f(\alpha^s$* cl $(A))$

But α^s* cl $(f(A))$ is the smallest α^s*-closed containing $f(A)$. This implies ,

$$\alpha^s\text{* cl } (f(A)) \subseteq f(\alpha^s\text{* cl } (A))$$

Claim $\tilde{f}(A)$ and $\tilde{f}(B)$ are adjacent if A and B are adjacent

It is enough to prove that α^s* cl $(f(A)) \cap \alpha^s$* cl $(f(B))$ is α^s*-open in (Y, τ_2).

Let A and B are adjacent in α^s* CLG (X, τ_1).

Since f is a bijection,

$$f(\alpha^s\text{* cl } (A) \cap \alpha^s\text{* cl } (B)) = f(\alpha^s\text{* cl } (A)) \cap f(\alpha^s\text{* cl } (B))$$
$$= \alpha^s\text{*cl } (f(A)) \cap \alpha^s\text{* cl } (f(B))$$

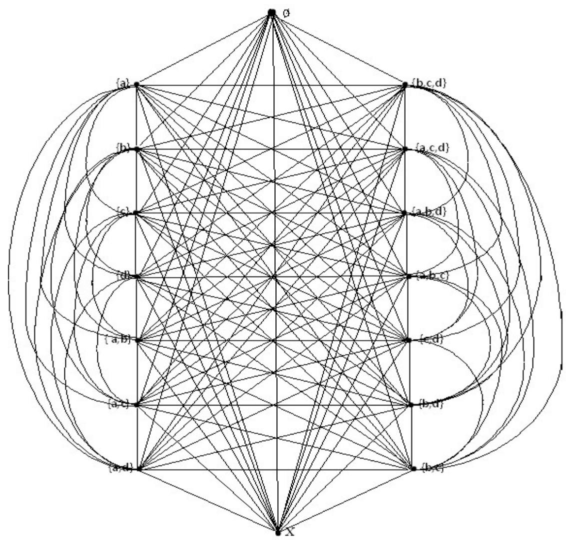
Since α^s* cl $(A) \cap \alpha^s$* cl (B) is α^s*-open in (X, τ_1) and f is a M-α^s*-homeomorphism, $f(\alpha^s$* cl $(A) \cap \alpha^s$* cl $(B))$ is α^s*-open in (Y, τ_2).

Therefore, α^s*cl $(f(A)) \cap \alpha^s$*cl $(f(B))$ is α^s*-open in (Y, τ_2).

Hence $\tilde{f}(A)$ and $\tilde{f}(B)$ are adjacent. Therefore $\tilde{f}$ is an isomorphism.

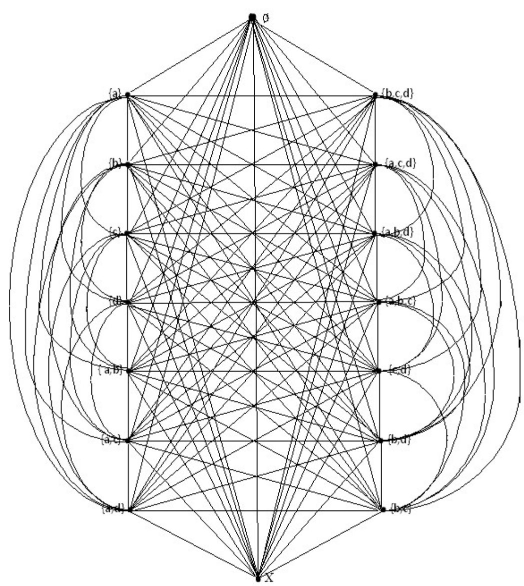
Remark 3.9. The converse of the above theorem is not true in the following example.

Consider the following example : Let $X = \{a, b, c, d\}$. Let $\tau_1 = \{\emptyset, \{a, b\}, \{c, d\}, X\}$. The α^s*-closure is



The above Graph α^s* CLG $(X, \tau_1) \cong \alpha^s$* CLG (Y, τ_2) but (X, τ_1) is not homeomorphic to (Y, τ_2).

Let $Y = X = \{a, b, c, d\}$. Let $\tau_2 = \{\emptyset, Y\}$. The α^s*-closure graph α^s* CLG (Y, τ_2) is given below.



Theorem 3.10. Let (X, τ) be a α^s*-connected topological space. A nonempty sets A and B are adjacent in α^s*-closure graph α^s* CLG (X, τ) then either A and B are α^s*-dense or disjoint.

Proof : Let A and B be two adjacent vertices in α^s* CLG (X, τ). Therefore α^s* cl $(A) \cap \alpha^s$* cl (B) is α^s*-open in (X, τ). Since (X, τ) is a α^s*-connected space, only $\emptyset$ and X are both α^s*-open and α^s*-closed in X.

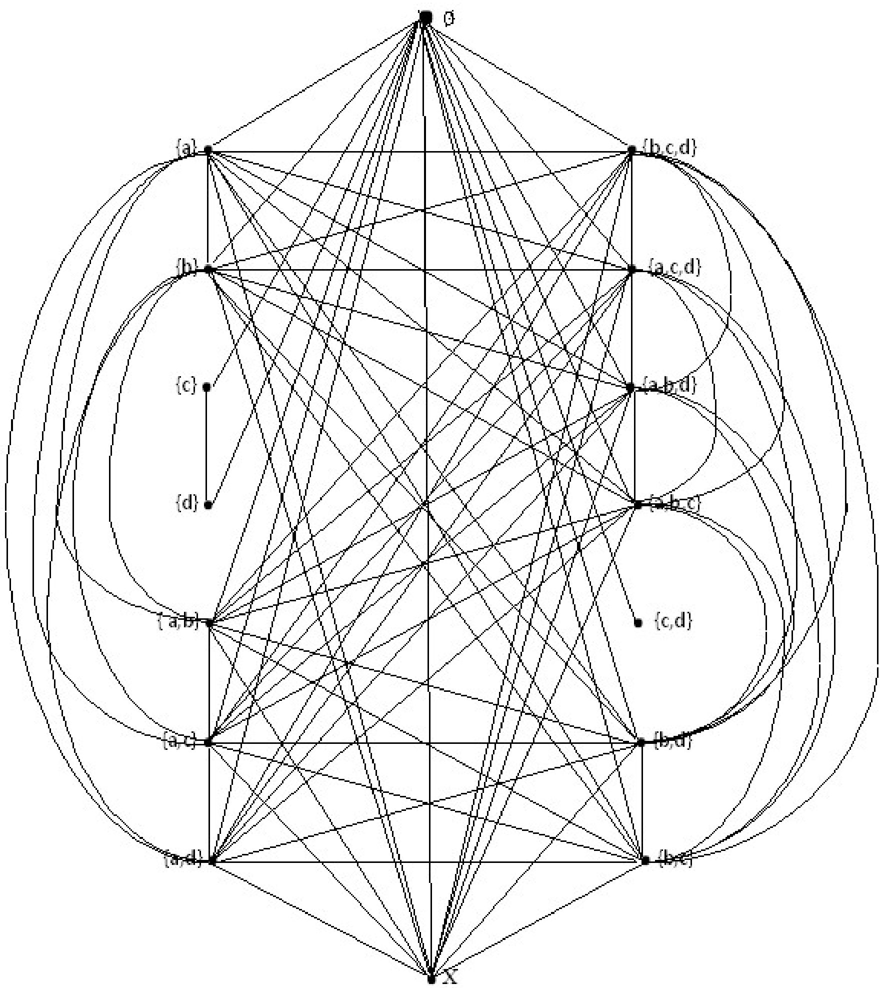
Case (i) : $\alpha^{s}*$ cl $(A) \cap \alpha^{s}*$ cl $(B) = X$.

Then $\alpha^{s}*$ cl $(A) = X$ and $\alpha^{s}*$ cl $(B) = X$. Therefore A and B are $\alpha^{s}*$-dense.

Case (ii) : $\alpha^{s}*$ cl $(A) \cap \alpha^{s}*$ cl $(B) = \emptyset$

Since $A \neq \emptyset$ and $B \neq \emptyset$, A and B are disjoint.

Remark 3.11. Converse of the above theorem is not true. Consider (X, τ) where $X = \{a, b, c, d\}$ $\tau = \{\emptyset, \{a, b\}, X\}$. $\{a\}$ and $\{d\}$ are disjoint but they are not adjacent



Theorem 3.12. Let (X, τ) be a $\alpha^{s}*$-connected space. Then the collection of non-empty nondisjoint $\alpha^{s}*$-closed sets is an independent set in $\alpha^{s}*$ *CLG* (X, τ).

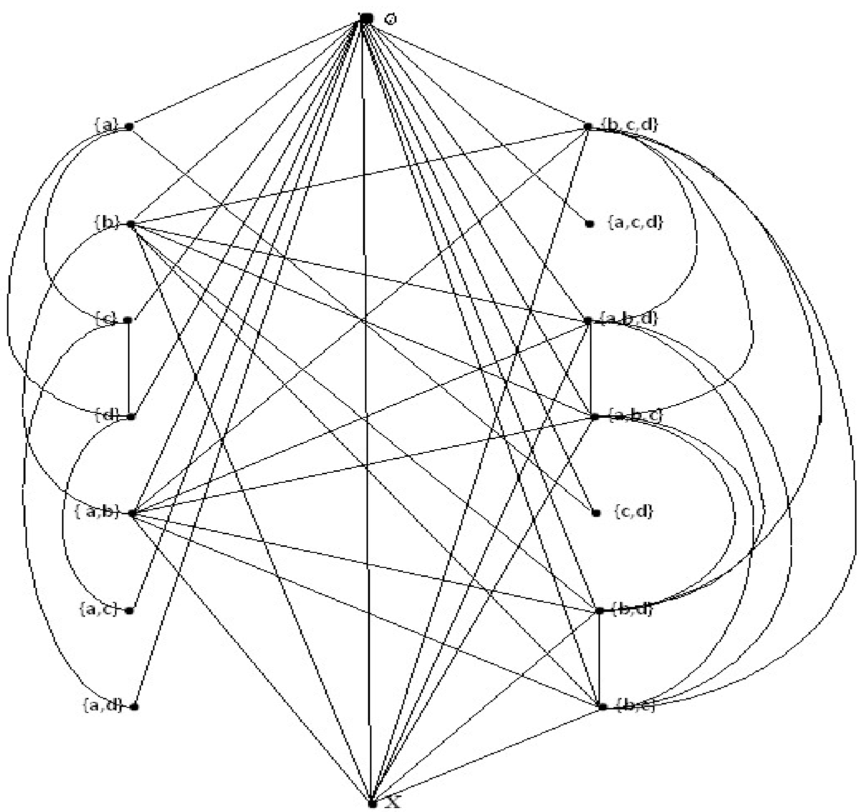
Proof : Let (X, τ) be a $\alpha^{s}*$-connected space. Let U be the collection of non-empty non-disjoint $\alpha^{s}*$-closed sets.

Let $A, B \in U$. If A and B are adjacent in $\alpha^{s}*$ *CLG* (X, τ), then $\alpha^{s}*$ cl $(A) \cap \alpha^{s}*$ cl (B) is $\alpha^{s}*$-open. Therefore $\alpha^{s}*$ cl $(A) \cap \alpha^{s}*$ cl (B) is both $\alpha^{s}*$-open and $\alpha^{s}*$-closed set other than X

and ∅. Which is a contradiction. Therefore A and B are non-adjacent and hence U is an independent in $\alpha^{s}*$ CLG (X, τ).

Remark 3.13. Converse of the above theorem is not true. Consider the following Example.

Consider (X, τ) where $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a, b\}, \{b\}, X\}$. Here $\{a, c, d\}$ and $\{b, c, d\}$ are independent sets. They are non-empty non-disjoint set but $\{b, c, d\}$ is not $\alpha^{s}*$-closed.



Theorem 3.14. Let (X, τ) be a $\alpha^{s}*$-connected topological space. Let $A \in P(X)$ be non-empty set. A is both $\alpha^{s}*$-open and $\alpha^{s}*$-dense in (X, τ) if and only if A is adjacent to some B and B^c in $\alpha^{s}*$ CLG (X, τ).

Proof : Let A be a nonempty set which is both $\alpha^{s}*$-open and $\alpha^{s}*$-dense. Take $B = \emptyset$ and $B^c = X$. Clearly A is adjacent to B. Also $\alpha^{s}* \text{ cl } (A) \cap \alpha^{s}* \text{ cl } (B^c) = X \cap X = X$. Therefore A is adjacent to B^c. Conversely A is adjacent to some B and B^c. That implies $\alpha^{s}* \text{ cl } (A) \cap \alpha^{s}* \text{ cl } (B)$ and $\alpha^{s}* \text{ cl } (A) \cap \alpha^{s}* \text{ cl } (B^c)$ are $\alpha^{s}*$-open sets. Therefore $[\alpha^{s}* \text{ cl } (A) \cap \alpha^{s}* \text{ cl } (B)] \cup [\alpha^{s}* \text{ cl } (A) \cap \alpha^{s}* \text{ cl } (B^c)]$ is $\alpha^{s}*$-open. Using Distributive Law $\alpha^{s}* \text{ cl } (A) \cap (\alpha^{s}* \text{ cl } (B) \cup \alpha^{s}* \text{ cl } (B^c))$ is $\alpha^{s}*$-open. That implies $\alpha^{s}* \text{ cl } (A)$ is $\alpha^{s}*$-open. Since X is $\alpha^{s}*$-connected topological space and A is non-empty $\alpha^{s}* \text{ cl } (A) = X$. Therefore A is $\alpha^{s}*$-dense. Therefore A is $\alpha^{s}*$-open and $\alpha^{s}*$-dense.

Theorem 3.14. Let (X, τ) be a topological space. If deg $(X) = 1$ in $\alpha^{s}*$ *CLG* (X, τ) then (X, τ) is a $\alpha^{s}*$-connected topological space.

Proof : Suppose X is not $\alpha^{s}*$-connected. Then (X, τ) has non-empty $\alpha^{s}*$-clopen set say A. clearly A is adjacent to X. That implies deg $(X) > 1$. Which is a contradiction to dex $(X) = 1$. Hence X is $\alpha^{s}*$-connected.

Remark 3.15. In example 3.2, (X, τ) is $\alpha^{s}*$-connected topological space but deg $(X)) = 4$ in $\alpha^{s}*$ *CLG* (X, τ).

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