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CONTACT CONFORMAL CONNECTION IN A TRANS –SASAKIAN MANIFOLD

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Oubina, J.A. [1] defined and initiated the study of Trans-Sasakian manifolds. Blair [2], Prasad and Ojha [3], Hasan Shahid [4] and some other authors have studied different properties of C-R-Sub –manifolds of Trans-Sasakian manifolds. Golab, S. [5] studied the properties of semi-symmetric and Quarter symmetric connections in Riemannian manifold. Yano, K. [6] has defined contact conformal connection and studied some of its properties in a Sasakian manifold. Mishra and Pandey [7] have studied the properties in Quarter symmetric metric F-connections in an almost Grayan manifold.

Result : In this paper we have defined and studied the contact conformal connection in a Trans-Sasakian manifold. Following the patterns of Yano [6], we have proved that if the curvature tensor of a contact conformal connection in an (α , 0) type Trans-Sasakian manifold vanishes, then the contact Bochner curvature tensor also vanishes.

Key words: Riemannian curvature tensor, Trans-Sasakian manifold, C-R-Sub –manifolds of Trans-Sasakian manifolds, semi-symmetric and Quarter symmetric connections in Riemannian manifold, almost Grayan manifold, contact Bochner curvature tensor.

INTRODUCTION

Let M_n $(n = 2m + 1)$ be an almost contact metric manifold endowed with a (1,1)-type structure tensor F, a contravariant vector field T, a –1 form A associated with T and a metric tensor 'g' satisfying :

$$F^2X = -X + A(X)T \quad ...(1.1a)$$

$$FT = 0 \quad ...(1.1b)$$

$$A(FX) = 0 \quad ...(1.1c)$$

$$A(T) = 1 \quad ...(1.1d)$$

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and $(\bar{X},\bar{Y}) = g(X, Y) - A(X)A(Y)$ ...(1.2a)

where $\bar{X} \stackrel{\text{def}}{=} FX$ ...(1.2b)

and $g(T, X) \stackrel{\text{def}}{=} A(X)$ ...(1.2c)

For all C^∞- vector fields X, Y in M_n also, a fundamental 2-form 'F in M_n is defined as

$$'F(X,Y) = g(\bar{X},Y) = -g(X,\bar{Y}) = -'F(Y,X) \quad ...(1.3)$$

Then, we call the structure bundle {F, T, A, g}an almost contact-metric structure [1]

An almost contact metric structure is called normal [1], if

$$(dA)(X,Y)T + N(X,Y) = 0 \quad ...(1.4a)$$

where

$$(dA)(X,Y) = (D_XA)(Y) - (D_YA)(X),$$

D is the Riemannian connection in M_n . ...(1.4b)

And $N(X, Y) = (D_{\bar{X}} F)(Y) - (D_{\bar{Y}} F)(X) - \overline{(D_X F)(Y)} + \overline{(D_Y F)(X)}$ ...(1.5)

Is Nijenhenus tensor in M_n .

An almost contact metric manifold M_n with structure bundle {F, T, A, g} is called a Trans-Sasakian manifold [3] & [1], if

$$(D_XF)(Y) = \alpha\{g(X,Y)T - A(Y)X\} + \beta\{'F(X,Y)T - A(Y)\bar{X}\} \quad ...(1.6)$$

where α , β are non -zero constants.

It can be easily seen that a Trans-Sasakian manifold is normal. In view of (1.6) one can easily obtain in M_n , the relations

$$N(X, Y) = 2\alpha 'F(X,Y)T \quad ... (1.7)$$

$$(dA)(X,Y) = -2\alpha 'F(X,Y) \quad ...(1.8)$$

$$(D_XA)(Y) + (D_YA)(X) = 2\beta\{g(X,Y) - A(Y)A(X)\} \quad ...(1.9)$$

$$(D_X 'F)(Y, Z) + (D_Y 'F)(Z, X) + (D_Z 'F)(X, Y) \quad ...(1.10)$$

$$= 2\beta[A(Z)'F(X,Y) + A(X)'F(Y, Z) + A(Y)'F(Z, X)]$$

$$(D_XA)(Y) = -\alpha 'F(X,Y) + \beta\{g(X,Y) - A(X)A(Y)\} \quad ...(1.11a)$$

$$(D_XT) = -\alpha\bar{X} + \beta\{X - A(X)T\} \quad ...(1.11b)$$

REMARK (1.1): In the above and in what follows, the letters X, Y, Z …..etc. an C^∞- vector fields in M_n .

CONTACT CONFORMAL CONNECTION IN A TRANS –SASAKIAN MANIFOLD M_n :

Let us consider a conformal connection of the metric tensor g which induces a new metric tensor $\tilde{g}$, given by

$$\tilde{g}(X,Y) = e^{2p}g(X,Y) \qquad ...(2.1)$$

With regard to this metric tensor $\tilde{g}$, we take an affine connection *B* which satisfies:

$$(B_X\tilde{g}) = B_X\{ e^{2p}g(X,Y)\} = 2e^{2p}\, p(X)A(Y)A(Z) \qquad ...(2.2)$$

where p is a C^∞ -Scalar point function in M_n .and

$$p(X) \overset{\text{def}}{=} g(P,X), \qquad ...(2.3)$$

being covariant derivative of the scalar *p* with respect to the metric tensor *g*,is a 1-form in M_n, where contra-variant associate vector field is *P*,further ,we assume that the torsion tensor of the connection B satisfies :

$$S(X,Y) = -2\ 'F(X,Y)U \qquad ...(2.4)$$

Where U is certain contra-variant vector field in M_n . In view of (2.2) and (2.4), we can easily obtain a relation between the connection B and the Riemannian connection D [6] , given by

$$B_YZ - D_YZ + \{Y-A(Y)T\}p(Z) + \{Z-A(Z)T\}p(Y) - g(\bar{Y},\bar{Z})P + u(Y)\bar{Z} + u(Z)\bar{Y} - 'F(Y, Z)U$$

where $u(X) \overset{\text{def}}{=} g(U,X)$

Now, suppose that *B* is an *F*-connection, then

$$(B_YF)(Z) = 0 = (D_YF)(Z) + \{Y-A(Y)T\}p(\bar{Z}) - p(Z)\bar{Y} + 'F(Y,Z)P + g(\bar{Y},\bar{Z})\bar{P} + u(\bar{Z})\bar{Y}$$
$$+u(Z)\{Y - A(Y)T\} - g(\bar{Y},\bar{Z})u + 'F(Y,Z)\bar{u}$$

Using (1.6), the above relation becomes

$$\alpha\{g(Y, Z)T - A(Z)Y\} + \beta\{'F(Y,Z)T - A(Z)\bar{Y}\} + p(\bar{Z})\{Y - A(Y)T\} - p(Z)\bar{Y} + 'F(Y,Z)P$$
$$+g(\bar{Y},\bar{Z})\bar{P} + u(\bar{Z})\bar{Y} + u(Z)\{Y-A(Y)T\} - g(\bar{Y},\bar{Z})u + 'F(Y,Z)\bar{u} = 0$$

Contracting the above equation with respect to Y, we have

$$-2m\alpha A(Z) + 2m\, p(\bar{Z}) - p(\bar{Z}) - p(\bar{Z}) + 2mu(Z) - u(Z) - u(Z) + 2A(U)A(Z) = 0$$

If we put

$$A(U) = u(T) = \alpha \qquad ...(2.7)$$

in (2.6), then we get

$$u(Z) = \alpha A(Z) - p(\bar{Z}) \qquad ...(2.8a)$$

or $U = \alpha T + \bar{P}$

Here, we take $\bar{P} = Q$ so that $q(Z) = g(Q,Z) = -p(\bar{Z})$ and $p(Q) = q(P) = 0$

Then (2.8) become

$$u(Z) = \alpha A(Z) + q(Z) \qquad ...(2.9a)$$

or $U = \alpha T + Q$ ...(2.9b)

Using the equation (2.9) in (2.5), we have

$$B_YZ = D_YZ + \{Y-A(Y)T\}p(Z) + \{Z- A(Z)T\}p(Y) - g(\bar{Y},\bar{Z})P$$
$$+\{ \alpha A(Y) + q(Y)\}\bar{Z}) + \{\alpha A(Z) + q(Z)\}(\bar{Y}) - 'F(Y, Z)\{ \alpha T + Q\}$$

Further, we suppose that *B* is a *T*–connection, then

$$B_Y T = 0 = D_Y T + p(T)\{Y - A(Y)T\}p(T) + \alpha\bar{Y} \quad ...(2.11)$$

Using (1.11)(b) in the above equation, we obtained

$$p(T) = A(P) = -\beta \quad ...(2.12)$$

Thus, we have

Proposition (1): In a Trans-Sasakian manifold M_n , the affine connection B which is an F–T-connection and whose torsion tensor satisfies (2.4), is given by (2.10), with the conditions

$$u(T) = \alpha = A(U)\ ;\ p(T) = A(P) = -\beta$$

and $$\bar{P} = Q\ ,\ q(Y) = -\,p(\bar{Y})$$

CURVATURE TENSOR OF THE CONTACT CONFORMAL CONNECTION

The curvature tensor of the contact conformal connection given by (2.10) is given by

$$R(X,Y,Z) = B_X B_Y Z - B_Y B_X Z - B_{[X,Y]}Z \quad ...(2.1.1)$$

Using (2.10), (2.12), (1.1),(1.2),(1.3), (1.6) and (1.11) in the above equation and after a straight forward computation, we obtained

$$R(X,Y,Z) = K(X,Y,Z) - \{X - A(X)T\}\ 'P(Y,Z) + \{Y - A(Y)T\}\ 'P(X,Z) \quad ...(2.1.2)$$
$$-g(\bar{Y},Z)P(X) + g(\bar{X},\bar{Z})P(Y) - 'Q(Y,Z)\bar{X} + 'Q(X,Z)\bar{Y} - 'F(Y,Z)Q(X)$$
$$+ 'F(X,Z)Q(Y) - 'F(X,Y)\bar{Z} - 'F(X,Y)w(Z)$$
$$+[(\alpha^2+\beta^2)\ 'F(Y,Z)\bar{X} - (\alpha^2+\beta^2)\ 'F(X,Z)\bar{Y} - 2\alpha^2\ 'F(X,Y)\bar{Z}]$$

where

$$'P(Y,Z) = (D_Y p)(Z) + \alpha^2 A(Y)A(Z) + \alpha A(Z)q(Y) + \alpha A(Y)q(Z) \quad ...(2.1.3a)$$
$$- p(Y)p(Z) + q(Y)q(Z) + \frac{1}{2}\,p(P)g(\bar{Y},\bar{Z})$$

$$P(Y) = D_Y P + \alpha^2 A(Y)T + \alpha q(Y)T + \alpha A(Y)Q - p(Y)P \quad ...(2.1.3b)$$
$$+ q(Y)Q + \frac{1}{2}\,p(P)\{Y - A(Y)T\}$$

$$'Q(Y,Z) = (D_Y q)(Z) - q(Z)p(Y) - q(Y)p(Z) - \alpha A(Z)p(Y) - \alpha A(Y)p(Z)$$
$$+ \frac{1}{2}p(P)\ 'F(Y,Z) \quad ...(2.1.4a)$$

$$Q(Y) = D_Y Q - q(Y)P - p(Y)Q - \alpha A(Y)p - \alpha p(Y)T + \frac{1}{2}\,p(P)\bar{Y} \quad ...(2.1.4b)$$

$$'V(X,Y) = -(dq)(X,Y) = -\{(D_X q)(Y) - (D_Y q)(X)\} \quad ...(2.1.5)$$

$$w(Z) = 2[p(Z)Q - q(Z)P + \beta A(Z)Q - \beta q(Z)T] \quad ...(2.1.6)$$

Since p(X) is a gradient vector, then

$$(D_X p)(Y) - (D_Y p)(X) = 0 \quad ...(2.1.7)$$

and consequently ‘P(X, Y) is symmetric, i.e.

$$'P(X, Y) = 'P(Y, X) \quad ...(2.1.8)$$

Also, we have p(T) = –β, then

$$(D_Y p)(T) + p(D_Y T) = 0$$

Or $$(DYp)(T) = -p[-\alpha\bar{Y} + \beta\{Y - A(Y)T\}]$$

$$= \alpha p(\overline{Y}) - \beta\, p(Y) - \beta^2 A(Y)$$

Or $$(D_Y p)(T) = -\alpha\, q(Y) - \beta\, p(Y) - \beta^2 A(Y) \quad ...(2.1.9)$$

Which is obtained by using (1.11)(b) and $q(Y) = -\, p(\bar{Y})$

Now, from (2.1.3)(a), we get

$$'P(Y, T) = (D_Y p)(T) + \alpha^2 A(Y) + \alpha q(Y) + \beta p(Y)$$

$$= -\alpha q(Y) - \beta p(Y) - \beta^2 A(Y) + \alpha^2 A(Y) + \alpha q(Y) + \beta p(Y)$$

Or $$'P(Y, T) = (\alpha^2 - \beta^2)A(Y) = 'P(T,Y) \quad ...(2.1.10)$$

From which, we get

$$'P(\bar{Y},T) = 0 = 'P(T, \bar{Y}) \quad ...(2.1.11)$$

Again, by taking covariant derivative of q(T) = 0 and using (1.11)(b), we obtain

$$(D_Y q)(T) = -\, q(D_Y p)(T) = -[\alpha\bar{Y} + \beta\{Y - A(Y)T\}] = \alpha\, q(\bar{Y}) - \beta q(Y)$$

$$= \alpha p(Y) + \alpha\beta A(Y) - \beta q(Y)$$

Using the above equation in (2.1.4)(a),we obtain

$$'Q(Y,T) = (D_Y q)(T) - \alpha p(Y) + \beta A(Y) + \beta q(Y)$$

$$= \alpha p(Y) + \alpha\beta A(Y) + \beta q(Y) - \alpha p(Y) + \alpha\beta A(Y) - \beta q(Y)$$

$$'Q(Y,T) = 2\,\alpha\beta A(Y) \quad ...(2.1.12)$$

Further, differentiating covariantly with respect to X, the expression

$$p(\bar{Z}) = -\, q(Z)$$

we have $$(D_Y p)(\bar{Z}) + p(D_Y F)(Z) + p(\overline{D_Y Z}) = -\,(D_Y q)(Z) - q(D_Y Z)$$

Using (1.6) here in the above equation, we get

$$(D_Y p)(\bar{Z}) + p[\alpha\{g(Y,Z)T - A(Z)Y\} + \beta\{'F(Y,Z)T - A(Z)\bar{Y}\}] = -\,(D_Y q)(Z)$$

Or

$$(D_Y p)(\bar{Z}) - \alpha\beta g(Y,Z) - \alpha A(Z)p(Y) - \beta^2\, 'F(Y,Z) + \beta A(Z)q(Y) = -\,(D_Y q)(Z)$$

Now, taking account of (2.1.3)(a) and (2.1.4)(a) in the above equation, we obtain

$$'P(Y, \bar{Z}) - \alpha A(Y)p(Z) - \alpha\beta A(Y)A(Z) - p(Y)q(Z) - q(Y)p(Z) - \beta A(Z)q(Y)$$

$$+ \tfrac{1}{2}p(P)\; 'F(Y,Z) - \alpha\beta g(Y,Z)$$

$$= -\alpha A(Z)p(Y) - \beta^2\, 'F(Y,Z) + \beta A(Z)q(Y)$$

$$= – ‘Q(Y,Z) – q(Z)p(Y) – q(Y)p(Z) – \alpha A(Z)p(Y) – \alpha A(Y)p(Z) + \frac{1}{2} p(P)\ ‘F(Y,Z)$$

Or $$‘P(Y, \bar{Z}) = – ‘Q(Y,Z) + \alpha\beta A(Y)A(Z) – \alpha A(Z)p(Y) + \beta A(Z)q(Y) + \alpha\beta g(Y,Z) + \alpha A(Z)p(Y) + \beta^2 ‘F(Y,Z) - \beta A(Z)q(Y)$$

Or

$$‘P(Y, \bar{Z}) = ‘Q(Y,Z) + \alpha\beta g(Y,Z) + \alpha\beta A(Y)A(Z) + \beta^2 ‘F(Y,Z) \quad ...(2.1.13)$$

From which by barring Z, using (1.1) and (2.1.10), we get

$$‘Q(Y, \bar{Z}) = ‘P(Y,Z) – \alpha\beta\ ‘F(Y,Z) + \beta^2 g(Y,Z) – \alpha^2 A(Y)A(Z) \quad ...(2.1.14)$$

Barring Y and using symmetricity of ‘P and (2.1.13), we obtain

$$‘Q(\bar{Y},\bar{Z}) = – ‘Q(Z,Y) + 2\alpha\beta g(Y,Z) \quad ...(2.1.15)$$ also, in view of (2.1.14), we get

$$‘Q(Y,\bar{Z}) – ‘Q(Z,\bar{Y}) = – 2\alpha\beta\ ‘F(Y,Z) \quad ...(2.1.16)$$

Further, putting Y = T in (2.1.13) and using (2.1.11) also (2.1.12), we have

$$‘Q(T, Z) = 2\alpha\beta A(Z) = ‘Q(Z,T) \quad ...(2.1.17a)$$

and $$‘Q(T,\bar{Z}) = – ‘Q(\bar{Z}, T) = 0 \quad ...(2.1.17b)$$

Now, putting Y = T in (2.1.5) and using (2.1.17)(a)

We can easily obtain

$$‘V(X,T) = 0 \quad ...(2.1.18)$$

Also from (2.1.6), we have

$$w(T) = 0, \quad ...(2.1.19a)$$

and $$‘w(Z,T) = g(w(Z), T) = 0 \quad ...(2.1.19b)$$

Now, from (2.1.5) and (2.1.6), we obtain

$$V^* = – 2D_{iv}P + 4m\beta^2 \quad ...(2.1.20a)$$

and $$w^* = 4\,(p(P) - \beta^2) \quad ...(2.1.20b)$$

where we have taken $V^* = F^{kj}V_{kj}$ and $w^* = F^{kj}w_{kj}$ then, we get

$$V^* – w^* = –2\{D_{iv}P + 2\,p(P) – 2(m+1)\beta^2\} \quad ...(2.1.21)$$

Now, taking account of ‘K(X, Y, Z, U) = ‘K(Z, U, X, Y)

where, ‘K(X,Y,Z,U) $\stackrel{\text{def}}{=}$ g (K(X, Y, Z), U) , we obtain from (2.1.2)

$$‘F(X,U)\{‘Q(Y,Z) + ‘Q(Z,Y)\} – ‘F(Y,U)\{‘Q(X,Z) – ‘Q(Z,X)\} + ‘F(Y,Z)\{‘Q(X,U) + ‘Q(U,X)\} – ‘F(X,Z)\{‘Q(Y,U) + ‘Q(U,Y)\} + ‘F(Z,U)\{V(X,Y) – ‘w(X,Y)\} – ‘F(X,Y)\{V(Z,U) – ‘w(Z,U)\} = 0 \quad ...(2.1.22)$$

From (2.1.22), we obtain, after a straight forward computation

$$‘Q(Y,Z) + ‘Q(Z,Y) = \frac{4\alpha\beta}{(m+1)} A(Y)A(Z) \quad ...(2.1.23)$$

Using this in (2.1.15), we have

$$'Q(\bar{Y},\bar{Z}) = 'Q(Y,Z) + 2\alpha\beta g(Y,Z) - \frac{4\alpha\beta}{(m+1)} A(Y)A(Z) \qquad ...(2.1.24)$$

Again using (2.1.23) in (2.1.22), we obtain, a after a straight forward computation

$$V(Z,U) - w(Z,U) = \frac{1}{2m} (V^* - w^*)'F(Z,U) \qquad ...(2.1.25)$$

And in consequence of (2.1.21), we also have

$$V(Z,U) - 'w(Z,U) = \frac{1}{2m} [D_{iv}P + 2p(P) - 2(m+1)\beta^2]\ 'F(Z,U) \qquad ...(2.1.26)$$

Now, from (2.1.4)(a) and (2.1.5), we obtain

$$'Q(Y,Z) - 'Q(Z,Y) = -V(Y,Z) + p(P)\ 'F(Y,Z) \qquad ...(2.1.27)$$

Using (2.1.23) in the above equation, we get

$$'V(Y,Z) = -2\ 'Q(Z,U) + p(P)\ 'F(Z,U) + \frac{4\alpha\beta}{(m+1)} A(Y)A(Z) \qquad ...(2.1.28)$$

So, in consequence of (2.1.28), (2.1.26) gives

$$'w(Z,U) = -2\ 'Q(Z,U) + p(P)\ 'F(Z,U) + \frac{4\alpha\beta}{(m+1)} A(Y)A(Z)$$

$$+ \frac{1}{m} [D_{iv}P + 2p(P) - 2(m+1)\beta^2]\ 'F(Z,U)$$

Or

$$'w(Z,U) = -2\ 'Q(Z,U) + \frac{1}{m} [D_{iv}P + (m+2)p(P) - 2(m+1)\beta^2]\ 'F(Z,U)$$

$$+ \frac{4\alpha\beta}{(m+1)} A(Z)A(U) \qquad ...(2.1.29)$$

From (2.1.3)(b) after contracting it, with respect to Y, we get

$$P^* = D_{iv}P + mp(P) + \alpha^2 - \beta^2 \qquad ...(2.1.30)$$

Then (2.1.29) becomes

$$'w(Z,U) = -2\ 'Q(Z,U) + \frac{1}{m} \{P^* + 2p(P) - \alpha^2 - (2m+1)\beta^2\}\ 'F(Z,U)$$

$$+ \frac{4\alpha\beta}{(m+1)} A(U)A(Z) \qquad ...(2.1.31)$$

Now, we suppose the curvature tensor with respect to the connection B vanishes, *i.e.*, R(X,Y,Z) = 0

Then from (2.1.2), we have

$$K(X,Y,Z) = \{X - A(X)T\}\ 'P(Y,Z) - \{Y - A(Y)T\}\ 'P(X,Z)$$

$$+ g(\bar{Y},\bar{Z})P(X) - g(\bar{X},\bar{Z})P(Y) + 'Q(Y,Z)\bar{X} - 'Q(X,Z)\bar{Y}$$

$$+ \,{}'F(Y, Z)Q(X) - {}'F(X, Z)Q(Y) + {}'B(B(X,Y))\bar{Z} + {}'F(X,Y)w(Z)$$

$$- [(\alpha^2 + \beta^2)\,{}'F(Y,Z)\bar{X} - (\alpha^2 + \beta^2)\,{}'F(X,Z)\bar{Y} - 2\alpha^2\,{}'F(X,Y)\bar{Z}] \quad ...(2.1.32)$$

Now, using (2.1.32) in

$${}'K(X, Y, Z, U) + {}'K(Y, Z, X, U) + {}'K(Z, X, Y, U) = 0$$

We obtain in consequence of the equation (2.1.14) , (2.1.23), (2.1.28) and (2.1.29)

$$+\frac{1}{m}[{}'F(Z, U)\,{}'F(X, Y) + {}'F(X, U)\,{}'F(Y, Z) + {}'F(Y, U)\,{}'F(X, Z)]$$

$$+[P^* + 2p(P) - \alpha^2 - 2(m+1)\beta^2 + mp(P) - 2m\,\beta^2] \quad ...(2.1.33)$$

$$+\frac{4\alpha\beta}{(m+1)}[A(U)A(Z)\,{}'F(X,Y) + A(X)A(U)\,{}'F(Y,Z) + A(Y)A(U)\,{}'F(X,Z)]$$

$$= 0$$

Barring U in the above equation, we get

$$P^* + (m+2)\,p(P) - \alpha^2 - \beta^2 = 0 \quad ...(2.1.34)$$

Using this in (2.1.31), we have

$${}'w(Z,U) = -2\,{}'Q(Z,U) - (\,p(P) + 2\,\beta^2)\,{}'F(Z,U) + \frac{4\alpha\beta}{(m+1)}A(U)A(Z) \quad ...(2.1.35)$$

Barring Y in (2.1.23), we have

$${}'Q(\bar{Y}, Z) + {}'Q(Z, \bar{Y}) = 0$$

Using it and (2.1.14) in (2.1.28), we get

$$V(\bar{Y}, Z) = 2\,{}'P(Y,Z) - 2\,\alpha^2 A(Y)A(Z) - 2\alpha\,\beta\,{}'F(Z,Y) + 2\,\beta^2 g(Z,Y)$$

$$- p(P)g(\bar{Y}, Z) \quad ...(2.1.36)$$

Also barring Z in (2.1.35), and using (2.1.14), we get

$${}'w(\bar{Z}, U) = 2\,{}'P(U,Z) - 2\,\alpha^2 A(U)A(Z) - 2\alpha\,\beta\,{}'F(U,Z) + 2\,\beta^2 g(U,Z)$$

$$- p(P)g(\bar{U}, Z) \quad ...(2.1.37)$$

Contracting with respect to the equation (2.1.32) and using (2.1.14), (2.1.36) and (2.1.37), we obtain

$$Ric(Y, Z) = 2(m+2)\,{}'P(Y,Z) - \{P^* - \alpha^2 + 3\beta^2\}A(U)A(Z)$$

$$+ \{P^* - 3\alpha^2 + 7\beta^2\}g(Y,Z) \quad ...(2.1.38a)$$

Or

$$K(Y) = 2(m+2)P(Y) - \{P^* + \alpha^2 + 3\beta^2\}A(Y)T$$

$$+\{P^* - \alpha^2 + 7\beta^2\}Y \quad ...(2.1.38b)$$

Contracting which with respect to Y, we get

$$k = 4(m+1)P^* - 2(3m+2)\,\alpha^2 + 2(7m+2)\beta^2$$

from which, we get

$$P^* = \frac{1}{4(m+1)}\ [k + 2(3m + 2)\alpha^2\ ] - \frac{7(m+2)}{2(m+1)}\beta^2 \qquad ...(2.1.38c)$$

$$= -\ L - \frac{7(m+2)}{2(m+1)}\beta^2$$

Where $L = -\frac{1}{4(m+1)}\ [k + 2(3m + 2)\alpha^2\ ]$

and k is scalar curvature

Now, using (2.1.38)(c) in (2.1.38)(a), we get

$$‘P(Y,Z) = -\ ‘L(Y,Z) - \frac{(m-4)\beta^2}{(4m+1)(m+2)}\ A(Y)A(Z)$$

$$- \frac{(7m+12)\beta^2}{(4m+1)(m+2)}\ g(Y,Z) \qquad ...(2.1.39a)$$

where $‘L(Y,Z) = -\frac{1}{2(m+2)}\ [\ Ric(Y,Z) + (L + 3\alpha^2)\ g(Y,Z)$

$- \ (L- \alpha^2)A(Y)A(Z)] \qquad ...(2.1.39b)$

Similarly, we obtain

$$‘Q(Y,Z) = ‘M(Y,Z) + \alpha\beta\ g(Y,Z) + \alpha\beta\ A(Y)A(Z)$$

$$+ \frac{(4m^2 + 5m - 4)\beta^2}{4(m+1)(m+2)}\ ‘F(Y,Z) \qquad ...(2.1.40a)$$

where $‘M(Y,Z) = -\frac{1}{2(m+2)}\ [Ric(\bar{Y},Z) + (L + 3\alpha^2)\ ‘F(Y,Z)] \qquad ...(2.1.40b)$

Now, from (2.1.38)(c) in (2.1.34), we get

$$p(P) = -\frac{1}{(m+2)}\ (L + \alpha^2)\ + \frac{(9m+4)\beta^2}{2(m+1)(m+2)} \qquad ...(2.1.41)$$

Now, using (2.1.41) , (2.1.40)(a) in (2.1.28)

$$V(X,Y) = 2\ ‘M(X,Y) + \frac{1}{(m+2)}\ (L + \alpha^2)F(X,Y) - \frac{2(m-2)\beta^2}{(m+2)}\ ‘F(X,Y)$$

$$- \frac{2\alpha\beta(m-1)}{(m+1)}\ A(X)A(Y) - 2\alpha\beta\ g(X,Y) \qquad ...(2.1.42)$$

Similarly we can obtain

‘ $$w(Z,U) = 2\ ‘M(Z,U) - \frac{1}{(m+2)}\ (L + \alpha^2)\ ‘F(Z,U) - \frac{(4m^2 + 12m + 4)\beta^2}{(m+1)(m+2)}\ ‘F(Z,U)$$

$$-\frac{2\alpha\beta(m-1)}{(m+1)}\,A(U)A(Z) - 2\alpha\beta\, g(Z,U) \quad ...(2.1.43)$$

Now, putting results (2.1.39)(a), (2.1.40)(a), (2.1.42) and (2.1.43) in (2.1.32) , we obtain

$$B^*(X,Y,Z) - \frac{(3m+8)\beta^2}{2(m+1)(m+2)}\{A(Y)A(Z)X - A(X)A(Z)Y + A(X)Tg(Y,Z) \quad ...(2.1.44)$$

$$- A(Y)Tg(X,Z)\} + \frac{(7m+12)\beta^2}{2(m+1)(m+2)}\{g(Y,Z)X - g(X,Z)Y\} - \alpha\beta\,\{g(Y,Z)\bar{X} - g(X,Z)\bar{Y}$$

$$+ `F(Y,Z)X - `F(X,Z)Y + A(Y)A(Z)\bar{X} - A(X)A(Z)\bar{Y} + A(X)T\, `F(Y,Z)$$

$$- A(Y)T\, `F(X,Z) - 2g(X,Y)\bar{Z} - 2\, `F(X,Y)\bar{Z} - 2A(Z)T\, `F(X,Y) - 2A(X)A(Y)\bar{Z}\}$$

$$-\frac{4\alpha\beta}{(m+1)}\{A(Z)T\, `F(X,Y) + A(X)A(Y)\bar{Z}\} - \frac{(2m^2+m-8)\beta^2}{2(m+1)(m+2)}\{`F(Y,Z)\bar{X} - `F(X,Z)\bar{Y}\}$$

$$+\frac{3m(2m-3)\beta^2}{(m+1)(m+2)}\, `F(X,Y)\bar{Z} = 0$$

where

$$B^*(X,Y,Z) = `K(X,Y,Z) + \{X - A(X)T\}\, `L(Y,Z) - \{Y - A(Y)T\}\, `L(X,Z)$$

$$+ g(\bar{Y},\bar{Z})L(X) - g(\bar{X},\bar{Z})L(Y) + `M(Y,Z)\bar{X} - `M(X,Z)\bar{Y} + `F(Y,Z)M(X)$$

$$- `F(X,Z)M(Y) - 2\{`F(X,Y)M(Z) + `M(X,Y)\bar{Z}\}$$

$$+ \alpha^2\{`F(Y,Z)\bar{X} - `F(X,Z)\bar{Y} - 2\, `F(X,Y)\bar{Z}\} \quad ...(2.1.45)$$

Here $B^*(X,Y,Z)$ is contact Bochner curvature in (α, 0) type Trans –Sasakian manifold so, we have

Theorem (2.1.1): Let M_n be an (α, 0) type Trans –Sasakian manifold equipped with contact conformal connection given by (2.10).

If the curvature tensor with respect to this connection vanishes then contact Bochner curvature also vanishes.

Proof: For (α, 0) type Trans –Sasakian manifold, β = 0, then from the equation (2.1.44), we easily obtain $B^*(X,Y,Z) = 0$

Remark (2.1): Further, if we take α = – 1 and β = 0, then M_n becomes Sasakian manifold and so in a Sasakian manifold equipped with contact conformal connection (2.1.28) we obtain the same result as in [6] obtain by K.Yano.

Further if we take α = 0, then

$$`L(Y, Z) = -\frac{1}{2(m+2)}\,[Ric(Y, Z) + Lg(\bar{Y}, \bar{Z})]$$

$$`M(Y, Z) = -\frac{1}{2(m+2)}\,[Ric(\bar{Y}, \bar{Z}) + L\, `F(Y, Z)]$$

Putting these results in (2.1.44), we have

$$K(X,Y,Z) - \frac{1}{2(m+2)} \text{Ric}(Y, Z)\{X - A(X)T\}$$

$$- \frac{1}{2(m+2)} \text{Lg}(\bar{Y}, \bar{Z}) \{X–A(X)T\} + \frac{1}{2(m+2)} \text{Ric}(X, Z)\{Y–A(Y)T\}$$

$$+ \frac{1}{2(m+2)} \text{Lg}(\bar{X}, \bar{Z})\{Y–A(Y)T\} - \frac{1}{2(m+2)} g(\bar{Y}, \bar{Z}) K(X)$$

$$- \frac{1}{2(m+2)} L\, g(\bar{Y}, \bar{Z})\{X–A(X)T\} + \frac{1}{2(m+2)} g(\bar{X}, \bar{Z})K(Y)$$

$$+ \frac{1}{2(m+2)} L\, g(\bar{X}, \bar{Z})\{Y - A(Y)T\} - \frac{1}{2(m+2)} \text{Ric}(\bar{Y}, \bar{Z})\bar{X}$$

$$- \frac{1}{2(m+2)} L\, {}^{‘}F(Y, Z)\bar{X} + \frac{1}{2(m+2)} \text{Ric}(\bar{X}, Z)\bar{Y} + \frac{1}{2(m+2)} L\, {}^{‘}F(X,Z)\bar{Y}$$

$$+ \frac{1}{(m+2)} [\, {}^{‘}F(X, Y)K(\bar{Z}) + L\, {}^{‘}F(X,Y)\, \bar{Z} + \text{Ric}(\bar{X}, Y)\bar{Z} + L\, {}^{‘}F(X,Y)\bar{Z}]$$

$$- \frac{(3m+8)\beta^2}{2(m+1)(m+2)} [A(Y)A(Z)X – A(X)A(Z)Y + A(X)Tg(Y,Z) – A(Y)T\, g(X,Z)]$$

$$+ \frac{(7m+12)\beta^2}{2(m+1)(m+2)} \{g(Y,Z)X – g(X,Z)Y\} - \frac{(2m-m-8)\beta^2}{2(m+1)(m+2)} \{\, {}^{‘}F(Y,Z)\bar{X} – {}^{‘}F(X,Z)\bar{Y}\}$$

$$+ \frac{3m(2m-m)\beta^2}{(m+1)(m+2)}\, {}^{‘}F(X,Y)\bar{Z} = 0 \quad ...(2.1.46)$$

Or

$$K(X,Y,Z) - \frac{1}{2(m+2)} [\text{Ric}(Y,Z)\{X\text{-}A(X)T\} – \text{Ric}(X,Z)\{Y – A(Y)T\}$$

$$+ g(\bar{Y},\bar{Z})K(X) – g(\bar{X},\bar{Z})K(Y) + \text{Ric}(\bar{Y}, Z)\bar{X} – \text{Ric}(\bar{X}, Z)\bar{Y} + {}^{‘}F(Y,Z)K(\bar{X})$$

$$– {}^{‘}F(X,Z)\, K(\bar{Y}) – 2\, {}^{‘}F(X,Y)\, K(\bar{Z}) – 2\, \text{Ric}(\bar{X}, Y)\bar{Z}\,]$$

$$+ \frac{k}{4(m+1)(m+2)} [g(\bar{Y},\bar{Z})\{X\text{- } A(X)T\} – g(\bar{X},\bar{Z})\{Y – A(Y)T\} + {}^{‘}F(Y,Z)\bar{X}$$

$$– {}^{‘}F(X,Z)\bar{Y} \text{ - } 2\, {}^{‘}F(X,Y)\bar{Z}] - \frac{(3m+8)\beta^2}{2(m+1)(m+2)} [A(Y)A(Z)X \text{ -}A(X)A(Z)Y$$

$$+ A(X)T\, g(Y,Z) – A(Y)T\, g(X,Z)] + \frac{(7m+12)\beta^2}{2(m+1)(m+2)} \{g(Y,Z)X – g(X,Z)Y\}$$

$$-\frac{(2m^2-m-8)\beta^2}{2(m+1)(m+2)}\{{}'F(Y,Z)\bar{X} - {}'F(X,Z)\bar{Y}\} + \frac{3m(2m-3)\beta^2}{(m+1)(m+2)}\ {}'F(X,Y)\bar{Z} = 0$$

Contacting above equation with respect to X, we get

$$\frac{6m\beta}{(m+2)}A(Y)A(Z) - \frac{6m\beta}{(m+2)}g(Y,Z) + \frac{6m\beta}{(m+2)}A(Y)A(Z) + \frac{m(7m+12)\beta^2}{(m+1)(m+2)}\ g(Y, Z)$$

$$-\frac{m(3m+12)\beta^2}{(m+1)(m+2)}\ A(Y)(Z) = 0$$

From which, we get

$$\beta\left[\frac{6m\beta}{(m+2)} - \frac{2m^2\beta}{(m+2)} + \frac{m(2m+1)(7m+12)\beta}{(m+1)(m+2)} + \frac{m(3m+8)\beta}{(m+1)(m+2)}\right] = 0$$

$\Rightarrow$ either $\beta = 0$, or $\beta = -\frac{3(m+1)\beta}{(6m+1)(m+2)}$

Theorem (2.1.2) : Let the curvature tensor with respect to contact conformal connection (2.10) vanishes. if$\alpha = 0$ then M_n is either Cosymplectic manifold or $(0, \beta)$ type Trans-Sasakian manifold with

$$\beta = -\frac{3(m+1)\beta}{(6m+1)(m+2)}$$

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