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LAPLACE TRANSFORM ON HOWELL'S SPACE–A NEW THEORY

V. N. MAHALLE [1], S.S. MATHURKAR [2], R. D. TAYWADE [3]

1. Assitt. Prof., Deptt. of Mathematics, Bar. R.D.I.K.N.K.D. College, Badnera Railway (M.S), India.

2. Assistant Prof., Dept. of Mathematics, Govt. College of Engineering, Amravati, (M.S), India

3. Assitt. Prof., Deptt. of Mathematics, Prof. Ram Meghe Institute of Technology & Research, Badnera, Amravati,

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In this paper we have discussed a new theory of Laplace transform defined on Howell's space. The definition of the Laplace transform on Howell's space of test functions is given. Also discussed some results and Fundamental theorems of Laplace Transform by using the same. Laplace Transform defined on Howell's space is linear, continuous and one-one mapping from $\boldsymbol{G}$ to $\boldsymbol{G}^c$ is also introduced.

Keywords: Laplace Transform, Howell's Space, Fundamental theorem.

INTRODUCTION

We know that any locally integrable function of exponential growth on $\boldsymbol{R}^n$ is Fourier transformable. A new theory of Fourier analysis developed by Howell, presented in an elegant series of papers [2-6] supporting the Fourier transformation of the functions on $\boldsymbol{R}^n$. Mahalle *et. al.* described a new theory of Mellin transform defined on Howell's space [7]. Bhosale [1] defined Fractional Fourier transform on Howell's space. Motivated by the above work, we are extending this theory to support Laplace transform. The main goal of the present work therefore is the development of a theory generalizing the well known theory of tempered distributions, which supports the Laplace transform of all exponentially bounded functions on $\boldsymbol{R}^n$.

In this paper, we propose a new theory of Laplace transform defined on Howell's space. In section II new notation and basic spaces are introduced. Fundamental theorem is described in section III. Definition of Laplace transform on Howell's space is given in Section IV. Some results are proved in section V and. The Laplace transform defined on this space is linear, continuous and one-one is stated in section VI. Lastly conclusion is given.

II. The Notations and Conventions:

2.1. Here we list some notations & terminologies from Howell's [2] and Zemanian [8] which will be used extensively in our work.

1. Powered Variable: Let $f: C^n \to C$, for each $\xi \in C^n$, $\varphi_\xi f$ will denoted the function f with powered variable by ξ given by $\varphi_\xi f|_t = f(t^\xi)$

2. Exponential Function: Let $\omega \in C^n$, where convenient e_ω will denote the function given by $e_\omega(t) = e^{\omega t}$

3. Translation operator: Let $f: C^n \to C$, for each $\xi \in C^n$, $\theta_\xi f$ will denote the translation of f by ξ given by $\theta_\xi f(t) = f(t - \xi)$

4. Scaling operator: For each $\alpha \in C$, $\sum f$ will denote the function f with the variable scaled by α, that is $\sum f|_t = f(\alpha\, t)$

5. Strips in C^n: For each $\alpha \geq 0$, B_α will denote subset of C^n given by $B_\alpha = \{ t =$ $t1,t2,\ldots tn \in Cn$:Im$tk \leq \alpha\,,\, k=1\,,\, 2\,,\ldots\,,n$

2.2 Basic spaces G and G^c:

Two spaces of functions G and G^c will be of especial importance. They are defined below:

2.2.1 Definition G Space:

A function $\varphi: C^n \to C$ is an element of G if and only if both of the following hold:

(1) φ is an analytic function of each complex variable.

(2) For every $\alpha > 0$, Sup $\left\{ e^{\alpha\, \rho(t)} |\, \varphi(t)\, | : t \in B_\alpha \right\} < \infty$.

2.2.2 Definition G^C Space:

A function $f: C^n \to C$ is an element of G^C if and only if both of the following hold:

(1) f is an analytic function of each complex variable.

(2) For every $\alpha > 0$, there is corresponding $\gamma \geq 0$ such that

Sup $\left\{ e^{-\gamma\, \rho\, (t)} |\, f(t)\, | : t \in B_\alpha \right\} < \infty$

G with a suitable topology will serve as the space of test functions. The following facts will be of importance and should all be obvious from the above definitions:

(1) G and G^C are both linear spaces closed under multiplication.

(2) $G \subset G^c$

(3) G contains all functions of the form $\varphi = \theta_\xi \eta_{\;\alpha}$, where $\alpha > 0$ and $\xi \in C^n$.

(4) If $\varphi \in G$ and $\xi, \zeta \in C^n$, then $e_{i\xi}\theta_\zeta \varphi$ is also in G and its restriction to R^n is in $L^1(R^n)$

(5) G^C contains all exponential functions on C^n.

(6) Any exponential function e_ξ with $\xi \in C^n$ is an element of G^C.

The topology on G is defined by the set of norms $\{ \|.\|_\alpha;\ \alpha \geq 0 \}$,

where $$\|\varphi\|_\alpha = \mathrm{Sup}\left\{e^{\alpha\,\rho(t)} |\,\varphi(t)\,| : t \in B_\alpha\right\}.$$

Any function φ, which is analytic in each variable is clearly in G if and only if $\|\varphi\|_\alpha$ is finite for every $\alpha > 0$. It is also obvious that if $\alpha \leq \beta$, then$\|\varphi\|_\alpha \leq \|\varphi\|_\beta$.

Similarly the topology on G^C is defined by the set of norms $\{ \|\,.\,\|_\alpha;\ \alpha \geq 0\}$

where $\|\varphi\|_\alpha = \mathrm{Sup}\left\{ e^{-\gamma\,\rho(t)} |\,\varphi(t)\,| : t \in B_\alpha\right\}$, for every $\alpha > 0$ there is a corresponding $\gamma \geq 0$.

G^C with suitable topology will serve as the space of test functions. It is also obvious that if $\alpha \leq \beta$then $\|\varphi\|_\alpha \leq \|\varphi\|_\beta$. Following lemma expressed some general relation.

2.3 Lemma:

Let $\alpha > 0, h \geq 0$ and $\varphi \in G$. Then for any $t \in B_\alpha$ and $\lambda \in C$ with $|\lambda| \leq \alpha$,

$\left|\varphi(t)\, e^{\lambda\,\rho(t)}\right| \leq \|\varphi\|_{\alpha+h} e^{-h\,\rho(t)}$.

Proof: Let t be any arbitrarily fixed elements of B_α and $\mu \in C$with $|\lambda| \leq \alpha$. Then

$$\|\varphi\|_{\alpha+h} = \mathrm{Sup}\left\{ e^{(\alpha+h)\,\rho\,(t)} |\varphi(t)\,| : t \in B_{\alpha+h}\right\}$$

$$\mathrm{Sup}\left\{ \left|e^{\lambda\,\rho\,(t)}\right| e^{h\,\rho(t)} \left|\,\varphi(t)\,\right| : t \in B_{\alpha+h}\right\}$$

$$\|\,\varphi(t)\,\|_{\alpha+h} e^{-h\,\rho(t)} \geq \left|\varphi(t)\, e^{\lambda\,\rho(t)}\right|$$

Therefore,

$$\left|\varphi(t)\, e^{\lambda\,\rho(t)}\right| \leq \|\varphi(t)\,\|_{\alpha+h} e^{-h\,\rho(t)}.$$

III. Fundamental Theorem:

Here we have explained fundamentally useful theorem.

3.1 Theorem: Let $\varphi \in G$ and $f \in G^c$ then the mapping $\varphi \to f\varphi$ is continuous linear mapping from G to G.

Proof: Since $f \in G^c$, therefore for all $\alpha \geq 0$ there exists $h > 0$ such that,

$$e^{-h\,\rho(t)} |\,f(t)\,| \leq I \quad \text{(Say)}$$

We have to show that $f\varphi \in G$.

$$\|f\varphi\|_\alpha = \mathrm{Sup}\left\{e^{\alpha\,\rho(t)} |f(t)\,\varphi(t)\,| : t \in B_\alpha\right\},$$

where B_α is as given in section II.

But $\varphi \in G$ then by lemma 2.3 with $\lambda = \alpha$, we have

$$\left|\, e^{\alpha\,\rho(t)} \varphi(t)\,\right| \leq e^{-h\,\rho(t)}\, \|\varphi\|_{\alpha+h} \tag{3.1}$$

Now from equation (3.1), we get

$$e^{\alpha\,\rho(t)}\,|\,f(t)\,|\,|\,\varphi(t)\,| \leq I\, \|\varphi\|_{\alpha+h}$$

Thus for any $\varphi \in G$,$\|f\varphi\|_\alpha \leq \infty$

It follows that $f\varphi$ is an element of G. So the mapping $\varphi \to f\varphi$ is continuous linear mapping from$G$to$G$.

IV. Definition:

Laplace transform on G: For each $f \in G$and $L(f)$ be the function on C^ngiven by,

$$[L(f)](s) = \int_{R^n} f(t)\ e^{-\mathrm{st}} d\Lambda_t\ ,$$

where Λ will denotes the Lebesgue measure on R^n.

This transform is well defined on Howell's space G, since G^ccontains all exponential functions on C^n that is $e^{-\mathrm{st}} \in G^c$, hence for $f(t) \in G, f(t)\, e^{-\mathrm{st}} \in G$ by fundamental theorem 3.1. Now by property (4) P.344 [2], restriction of $f(t)\, e^{-\mathrm{st}} \in G$ to R^nbelongs to $L^1(R^n)$. Hence there exists right hand side.

V. Elementary Laplace Analysis on G:

A preliminary development of Laplace analysis on G is given and for this we have proved following results:

5.1 Result:

Let β be any non-zero real number and let f be any element of G. Then

$$\left[L\left(\sum{}_{\beta} f\right)\right](s) = |\beta|^{-n}\,[L(f)]\left(\frac{s}{\beta}\right)$$

5.2 Result:

Let $f \in G$. Then for any ξ and β in C^n,

$$\left[L\left(e_{\xi} \sum{}_{\beta} f\right)\right](s) = |\beta|^{-n}\left[L\left(e_{\xi\beta^{-1}} f\right)\right]\left(\frac{s}{\beta}\right)$$

5.3 Result:

For $f \in G$ and any $\omega \in C^n$, $[L(e_\omega f)]\ (s) = \theta_\omega [L(f)\ ]\ (s)$

5.4 Result:

If $f \in G$, then $\psi = L(f)$ is an analytic function of each variable and for $k = 1, 2, 3, - - - -, n$

(i) $$\frac{\partial}{\partial \sigma_k}\ \psi(\sigma + i\zeta) = L[(-t_k)f]\ (\sigma + i\zeta)$$

(ii) $$\frac{\partial}{\partial \zeta_k} \psi\ (\sigma + i\zeta) = L[(-\mathrm{it}_k)f]\ (\sigma + i\zeta)$$

(iii) $$\frac{\partial}{\partial s_k} \psi\ (s) = L[(-t_k)f]\ (s)$$

VI. Continuity of $L(f)$ and Other Operators:

Here we prove some theorems for continuity of $L(f)$ in G^c.

6.1 Theorem:

For each $f \in G$, the mapping $\psi: f \to L(f)$ is a continuous, linear and one-one mapping from G to G^c.

Proof: Let $f \in G$and $\psi = L(f)$. To show that ψ is in G^c. We have to show that $\|\psi\|_\alpha$ is bounded for suitable $\gamma \geq 0$. Let$s$ be any element of B_α.

$$\|\psi\|_\alpha = \text{Sup}\left\{e^{-\gamma\,\rho\,(s)} |\, \psi(s)\,| : s \in B_\alpha\right\}$$

Now,

$$\left|e^{-\gamma\,\rho\,(s)} |\, \psi(s)\,|\right| = \left|e^{-\gamma\,\rho\,(s\,)} |\,[L(f)]\,(s)|\,\right|$$
$$\left|e^{-\gamma\,\rho\,(s\,)}\right| \int_{R^n} |f(t)\, e^{-\mathrm{st}}| d\Lambda_t$$
$$e^{-\gamma\,\rho\,(s)} \int_{R^n} M e^{-\alpha\, n\, t}\, d\Lambda_t, \forall \alpha > 0$$
$$\frac{M}{\alpha\, n} e^{-\gamma\,\rho\,(s)}$$
$$\infty, \gamma \geq 0$$

Therefore $\psi: f \to L(f)$is continuous, linear and one-one mapping from $G$to G^c.

6.2 Theorem:

For each non-zero real valueβ, $\sum_\beta$ is a continuous, linear and one-one mapping from $G$to $G^c$with inverse $\sum_{\beta^{-1}}$ satisfying the following equality. For each $f \in G$and $\psi(t) = e^{-\mathrm{st}} \in G^c$

$$\int_{R^n} \sum{}_\beta f\psi\, d\Lambda = |\beta|^{-n} \int_{R^n} f\left[\sum{}_{\beta^{-1}} \psi\right] d\Lambda$$

Proof: The first part is easy and hence proof is not given here.

Now we prove second part,

Consider,

$$\int_{R^n} \sum{}_\beta f\psi\, d\Lambda = \int_{R^n} \sum{}_\beta f(t)\, e^{-st} d\Lambda_t$$
$$= |\beta|^{-n} \int_{R^n} f(u)\;\; e^{-s\frac{u}{\beta}} d\Lambda_u$$
$$= |\beta|^{-n} \int_{R^n} f\left[\sum{}_{\beta^{-1}} \psi\right] d\Lambda$$

6.3 Theorem:

For each $\omega \in C^n$ and $\theta_\omega$is a linear, continuous and one-one mapping from G to G^c with inverse $\theta_{-\omega}$. Moreover for each$f \in G$ and $\psi(t) = e^{-\mathrm{st}} \in G^c, \theta_\omega$ satisfying the following equality $\int_{R^n} [\theta_\omega f]\;\, \psi\, d\Lambda = \int_{R^n} f[\theta_{-\omega}\psi]\;\, d\Lambda$.

Proof : First part proof is so simple, so not given here.

Now consider,

$$\int_{R^n} [\theta_\omega f]\ \psi\, d\Lambda = \int_{R^n} f(t-\omega)\ \psi\, d\Lambda_t$$
$$= \int_{R^n} f(u)\ e^{-s\,(u+\omega)}\, d\Lambda_u$$
$$= \int_{R^n} f[\theta_{-\omega}\psi]\ d\Lambda$$

VII. Conclusion:

In the present paper we have introduced a new theory of Laplace transform defined on Howell's space. By using notations and terminology found some results. Proved Fundamental theorem and also proved that Laplace transform defined on Howell's space is linear, continuous and one-one mapping from GtoG^c.

REFERENCES

1. B. N. Bhosale : Integral Transformations of Generalized Functions (2005).
2. K. B. Howell : A New Theory for Fourier Analysis Part I, The Space of Test Functions, Journal of Mathematical Analysis and Applications, **168**, pp.342-350 (1992).
3. K. B. Howell : A New Theory for Fourier Analysis Part II, Further Analysis on the Space of Test Functions, Journal of Mathematical Analysis and Applications, **173**, pp.419-429 (1993).
4. K. B. Howell : A New Theory for Fourier Analysis Part III, Basic Analysis on the Dual, Journal of Mathematical Analysis and Applications, **175**, pp.257-267 (1993).
5. K. B. Howell : A New Theory for Fourier Analysis Part IV, Basic Multiplication and Convolution on the Dual Space, Journal of Mathematical Analysis and Applications, **180**, pp.79-92 (1993).
6. K.B. Howell : A New Theory for Fourier Analysis Part V, Generalized Multiplication and Convolution on the Dual Space, Journal of Mathematical Analysis and Applications,1**87**, pp.567-582 (1994).
7. Mahalle V. N. and *et. al.* : Mellin Transform on Howell's Space- A New Theory, Acta Ciencia Indica, Vol.XXXVI M, No.4, pp.629-634 (2010).
8. A. H. Zemanian : Genaralized Integral Transformation, Inter Science Publishers, New York (1968).

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