Acta Ciencia Indica, Vol. XL M, No. 2, 239 (2014)

RANDOM FIXED POINTS OF RANDOM MULTIVALUED OPERATORS ON POLISH SPACE

SHOYEB ALI SAYYED

Principal, Royal College of Technology, Indore (M.P.), India

LATA VYAS

Lecture, Deptt. of Mathematics, Lakshmi Narain College of Technology, Indore (M.P.), India

AND

FARKHUNDA SAYYED

Professor, Deptt. of Mathematics, Lord Krishna College of Technology, Indore (M.P.), India

RECEIVED : 22 January, 2014

REVISED : 22 March, 2014

In this paper we have proved a random fixed point theorem for a pair of two random multivalued operators with generalized contraction.

KEYWORDS AND PHRASES: Polish space, random multivalued, operator, random fixed point, measurable mappings, Hausdorff metric.

AMS (2010) Subject Classification: Primary 47H10; Secondary 54H25, 47H40, 47H04.

INTRODUCTION

Random fixed point theory has received much attention in recent years and it is needed for the study of various classes of random equations. The study of random fixed point theorems was initiated by the Prague school of probabilistic in the 1950s.The interest in this subject enhanced after publication of the survey paper of Adomain [1], Bharuch-Reid [2], Itoh [8] and Fan [7].

In polish spaces, random fixed point theorems for contraction mappings were proved by Sonodia, *et. al* [11].

In this direction Mehta and Dhagat [9], Shrivastava, *et. al* [12] proved some fixed point theorems in polish spaces. Chouhan [5] established common fixed point theorem for four random operator in Hilbert space. Beg *et al*. [3] obtained random version in convex separable complete metric spaces and Nashine [10] proved new random fixed point results for generalized altering distance functions. Recently Dhage *et al.* [6] proved fixed point theorems for a pair of generalized weakly contractive mapping in polish space.

PRELIMINARIES

Let (X, d) be a polish space that is separable complete metric space and (Ω, α) be measurable space. Let 2^x be a family of all subsets of x and $CB(X)$ denote the family of all nonempty bounded closed subsets of *X*. *A* mappings $T:\Omega \to 2^x$ is called measureable if for any open subset *C* of *X*, $T^{-1}(C)=\{w\in\Omega : T(w)\cap C\neq\phi\}\in\alpha$. A mapping $\xi:\Sigma\to X$ is said to be measurable sector of a measurable mapping $T:\Omega\to 2^x$, if ξ is measurable and for any $w\in\Omega, \xi(w)\in T(w)$. A mapping $f:\Omega xX\to X$ is called random operator if for any $x\in X, f(:,x)$ is measurable. A mapping $T:\Omega xX\to CB(X)$ is a random multivalued operator, if for ever $x\in X$, $T(:,x)$ is measurable. A measurable mapping $\xi:\Omega\to X$ is called random fixed point of a random multivalued operator $T:\Omega xX\to CB(X)(f:\Omega xX\to X)$. If for every $w\in\Omega,\xi(w)\in T(w,\xi(w))(f(w,\xi(w))=\xi(w))$. Let $T:\Omega xX\to CB(X)$ be a random operator and $\{\xi_n\}$ is said to be asymptotically T-regular, if $d(\xi_n(w),T(w,\xi_n(w)))\to 0$.

COMMON RANDOM FIXED POINT FOR MULTIVALUED OPERATIONS WITH GENERALIZED CONTRACTION

Theorem 3.1: Let *X* be a polish space. Let *T*, *S* : Ω *x* *X* $\to$ *CB*(*X*) be tow continuous random multivalued operators. If there exists a measurable mapping $\alpha:\Omega\to(0,1)$, such that $H(S(\omega,x),T(\omega,y))\le\alpha(\omega)\,[d(x,S(\omega,x))+d(y,T(\omega,y))]$

For each $x, y\in X$, $\omega\in\Omega$ and α is non-negative with $\alpha(\omega)<\frac{1}{2}$, then there exists a common random fixed point of *S* and *T* (here *H* represents the Hausdorff metric on *CB*(*X*) induced by metric *d*).

Proof: Let $\xi_0:\Omega\to X$ be an arbitrary measurable mapping and choose a measurable mapping $\xi_1:\Omega\to X$ such that $\xi_1(\omega)\in S(\omega,\xi_0(\omega))$ for each $\omega\in\Omega$. Then for each $\omega\in\Omega$,

$$H(S(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega)))\le\alpha(\omega)[d(\xi_0(\omega),S(\omega,\xi_0(\omega)))+d(\xi_1(\omega),T(\omega,\xi_1(\omega)))]$$

It further implies [4, lemma 2.3], then there exists a measurable mapping $\xi_2:\Omega\to X$ such that for any $\omega\in\Omega$, $\xi_2(\omega)\in T(\omega,\xi_1(\omega))$ and

$$d(\xi_1(\omega),\xi_2(\omega))\le H(S(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega)))$$

$$d(\xi_1(\omega),\xi_2(\omega))\le\alpha(\omega)[d(\xi_0(\omega),\xi_1(\omega)),+d(\xi_1(\omega),\xi_2(\omega))]$$

$$d(\xi_1(\omega),\xi_2(\omega))\le\frac{\alpha(\omega)}{1-\alpha(\omega)}d(\xi_0(\omega),\xi_1(\omega))$$

$$d(\xi_1(\omega),\xi_2(\omega))\le k\ d(\xi_0(\omega),\xi_1(\omega)),\text{ where } k=\frac{\alpha(\omega)}{1-\alpha(\omega)}<1.$$

By above lemma 2.3 of [4] in the same manner, there exists a measurable mapping $\xi_3:\Omega\to X$, such that for any $\omega\in\Omega$, $\xi_3(\omega)\in S(\omega,\xi_2(\omega))$ and

$$\begin{aligned}d(\xi_2(\omega),\xi_3(\omega))&\le H(T(\omega,\xi_1(\omega)),S(\omega,\xi_2(\omega)))\\&\le\alpha(\omega)[d(\xi_1(\omega),T(\omega,\xi_1(\omega)))+d(\xi_2(\omega),S(\omega,\xi_2(\omega)))]\\&\le\alpha(\omega)[d(\xi_1(\omega),\xi_2(\omega))+d(\xi_2(\omega),\xi_3(\omega))]\end{aligned}$$

$$d(\xi_2(\omega),\xi_3(\omega)) \le \frac{\alpha(\omega)}{1-\alpha(\omega)} d(\xi_1(\omega),\xi_2(\omega))$$

$$d(\xi_2(\omega),\xi_3(\omega)) \le k^2 d(\xi_0(\omega),\xi_1(\omega)).$$

Similarly, proceeding in the same way by induction method, we produce a sequence of measurable mapping $\xi_n : \Omega \to x$ such that for $h > 0$ and any $\omega \in \Omega$,

$$\xi_{2h+1}(\omega) \in S(\omega,\xi_{2h}(\omega)), \xi_{2h+2}(\omega) \in T(\omega,\xi_{2h+1}(\omega))$$

and $$d(\xi_n(\omega),\xi_{n+1}(\omega)) \le kd(\xi_{n-1}(\omega),\xi_n(\omega)) \le .... \le k^n d(\xi_0(\omega),\xi_1(\omega)).$$

Furthermore for $m > n$

$$d(\xi_n(\omega),\xi_m(\omega)) \le d(\xi_n(\omega),\xi_{n+1}(\omega)) + d(\xi_n + 1(\omega),\xi_{n+2}(\omega)) + ... + d(\xi_{m-1}(\omega),\xi_m(\omega))$$

$$\le \{k^n + k^{n+1} + ... + k^{m-1}\} d(\xi_0(\omega),\xi_1(\omega))$$

$$d(\xi_n(\omega),\xi_m(\omega)) \le \frac{k^n}{1-k} d(\xi_0(\omega),\xi_1(\omega)) \to 0 \text{ as } n, m \to \infty.$$

It follows that $\{\xi_n(\omega)\}$ is a Cauchy sequence and there exists a measurable mapping $\xi : \Omega \to X$ such that $\xi_n(\omega) \to \xi(\omega)$ for each $\omega \in \Omega$. It further implies that $\xi_{2h+1}(\omega) \to \xi(\omega)$ and $\xi_{2h+2}(\omega) \to \xi(\omega)$. Thus we have for any $\omega \in \Omega$.,

$$d(\xi(\omega), S(\omega,\xi(\omega))) \le d(\xi(\omega),\xi_{2h+2}(\omega)) + d(\xi_{2h+2}(\omega), S(\omega,\xi(\omega)))$$

$$\le d(\xi(\omega),\xi_{2h+2}(\omega)) + [H(T(\omega,\xi_{2h+1}(\omega)), S(\omega,\xi(\omega)))]$$

$$\le d(\xi(\omega),\xi_{2h+2}(\omega)) + \alpha(\omega)[d(\xi_{2h+1}(\omega),$$
$$T(\omega,\xi_{2h+1}(\omega))) + d(\xi(\omega), S(\omega,\xi(\omega)))]$$

$$\le d(\xi(\omega),\xi_{2h+2}(\omega)) + \alpha(\omega)[d(\xi_{2h+1}(\omega), \xi_{2h+2}(\omega)) + d(\xi(\omega), S(\omega,\xi(\omega)))]$$

Letting $h \to \infty$, we have

$$d(\xi(\omega), S(\omega,\xi(\omega))) \le \alpha(\omega) d(\xi(\omega), S(\omega,\xi(\omega))).$$

Hence $\xi(\omega) \in S(\omega,\xi(\omega))$ for $\omega \in \Omega$. Similarly, for any $\omega \in \Omega$,

$$d(\xi(\omega), T(\omega,\xi(\omega))) \le d(\xi(\omega),\xi_{2h+1}(\omega)) + d(\xi_{2h+1}(\omega), T(\omega,\xi(\omega)))$$

$$\le \alpha(\omega) d(\xi(\omega), T(\omega,\xi(\omega))).$$

Therefore $\xi(\omega) \in T(\omega, \xi(\omega))$ for each $\omega \in \Omega$.

Corollary 3.2: Let X be a polish space and $T : \Omega\ x\ X \to CB(X)$ be a continuous random multivalued operator. If there exists a measurable mapping $\alpha : \Omega \to (0,1)$ such that $\omega \in \Omega$, $H(T(\omega,x), T(\omega,y)) \le \alpha(\omega)[d(x, T(\omega,x)) + d(y, T(\omega,y))]$ for each x, $y \in X$, and is non-negative with $\alpha < \frac{1}{2}$ then there exists a sequence $\{\xi_n\}$ of measurable mappings $\xi_n : \Omega \to X$ which is asymptotically T-regular and converges to a random fixed point of T.

COMMON RANDOM FIXED POINT OF PAIR OF MULTIVALUED OPERATORS

Theorem 4.1: Let X be a polish space, $T : \Omega\ x\ X \to CB(X)$ and $S : \Omega x X \to CB(X)$ be two continuous random operators satisfying (1). If there exists an asymptotically T-regular and S-regular sequences $\{\xi_n\}$ of measurable mappings $\xi_n : \Omega \to x$, then there exists a measurable mapping $\xi : \Omega \to X$ such that for any $\omega \in \Omega$, $\xi(\omega) \in T(\omega, \xi(\omega))$ and $\xi(\omega) \in S(\omega, \xi(\omega))$.

Moreover $T(\omega, \xi_n(\omega)) \to T(\omega, \xi(\omega))$ and $S(\omega, \xi_n(\omega)) \to S(\omega, \xi(\omega))$.

Proof: By previous theorem and induction method, we produce a sequence of measurable mapping $\xi_n : \Omega \to X$ such that for $m > 0$ and any $\omega \in \Omega$

$$\xi_{m+1}(\omega) \in S(\omega, \xi_m(\omega)), \xi_{m+2}(\omega) \in T(\omega, \xi_{m+1}(\omega))$$

and
$$d(\xi_n(\omega), \xi_{n+1}(\omega)) \leq kd(\xi_{n-1}(\omega), \xi_n(\omega)) \leq .... \leq k^n d(\xi_0(\omega), \xi_1(\omega)).$$

Furthermore, for $m > n$,

$$d(\xi_n(\omega), \xi_m(\omega)), \leq d(\xi_n(\omega), \xi_{n+1}(\omega)) + d(\xi_{n+1}(\omega), \xi_{n+2}(\omega)) + ... + d(\xi_{m-1}(\omega), \xi_m(\omega))$$
$$\leq [k^n + k^{n-1} + ... + k^{m-1}] d(\xi_0(\omega), \xi_1(\omega)).$$

It follows that $\{\xi_n(\omega)\}$ is a Cauchy sequence and there exists a measurable mapping $\xi : \Omega \to X$ such that $\xi_n(\omega) \to \xi(\omega)$ for each $\omega \in \Omega$. It further implies that $\xi_{m+1}(\omega) \to \xi(\omega)$ and $\xi_{m+2}(\omega) \to \xi(\omega)$. That we have for any $\omega \in \Omega$,

$$d(\xi_n(\omega), S(\omega, \xi_n(\omega) \leq d(\xi_n(\omega), \xi_{m+2}(\omega)) + d(\xi_{m+2}(\omega), S(\omega, \xi_n(\omega)))$$
$$\leq (\xi_n(\omega), \xi_{m+2}(\omega)) + H(T(\omega, \xi_{m+1}(\omega)), S(\omega, \xi_n(\omega)))$$
$$\leq d(\xi_n(\omega), \xi_{m+2}(\omega)) + \alpha(\omega)[d(\xi_{m+1}(\omega), T(\omega, \xi_{m+1}(\omega))) + d(\xi_n(\omega), S(\omega, \xi_n(\omega)))$$
$$\leq d(\xi_n(\omega), \xi_{m+2}(\omega)) + \alpha(\omega)[d(\xi_{m+1}(\omega), \xi_{m+2}(\omega)) + d(\xi_n(\omega), S(\omega, \xi_n(\omega)))].$$

Letting $m \to \infty$, we have

$$d(\xi_n(\omega), S(\omega, \xi_n(\omega)) \leq \alpha(\omega) d(\xi_n(\omega), S(\omega, \xi_n(\omega))).$$

Hence $\xi_n(\omega) \in S(\omega, \xi_n, (\omega))$ for $\omega \in \Omega$.

Similarly, for any $\omega \in \Omega$,

$$d(\xi_n(\omega), T(\omega, \xi_n(\omega))) \leq d(\xi_n(\omega), \xi_{m+1}(\omega)) + d(\xi_{m+1}(\omega), T(\omega, \xi_n(\omega)))$$
$$\leq d(\xi_n(\omega), \xi_{m+1}(\omega)) + H(S(\omega, \xi_m(\omega)), T(\omega, \xi_n(\omega)))$$
$$\leq \alpha(\omega) d(\xi_n(\omega), T(\omega, \xi_n(\omega))).$$

Therefore $\xi_n(\omega) \in T(\omega, \xi_n(\omega))$ for each $\omega \in \Omega$. Now

$$H(S(\omega, \xi_n(\omega)), T(\omega, \xi_m(\omega)) \leq d(S(\omega, \xi_n(\omega)), \xi_n(\omega)) +$$
$$d(\xi_n(\omega), \xi_m(\omega)) + d(\xi_m(\omega), T(\omega, \xi_m(\omega))).$$

Thus $\{T(\omega, \xi_n(\omega))\}$ and $\{S(\omega, \xi_n(\omega))\}$ are Cauchy sequences in $CB(X)$, therefore there exists $A(\omega) \in CB(X)$ such that $H(S(\omega, \xi_n(\omega)), A(\omega)) \to 0$ (By Itoh [81, Proposition 1], $A : \Omega \to CB(X)$ is measurable). Let $\xi : \Omega \to X$ be a measurable mapping such that for each $\omega \in \Omega, \xi(\omega) \in A(\omega)$ then for any $\omega \in \Omega$,

$$d(\xi(\omega), T(\omega, \xi(\omega)) \leq H(A(\omega), T(\omega, \xi(\omega)))$$

$$= \lim_{n\to\infty} H(S(\omega, \xi_n(\omega)), T(\omega, \xi(\omega)))$$

$$\leq \lim_{n\to\infty} \alpha(\omega), [d(\xi_n(\omega)) S(\omega, \xi_n(\omega))) + d(\xi(\omega), T(\omega, \xi(\omega)))]$$

$$\leq \lim_{n\to\infty} \alpha(\omega)[d(\xi_n(\omega), \xi_n(\omega)) + d(\xi(\omega), T(\omega, \xi(\omega)))]$$

or $\quad d(\xi(\omega), T(\omega, \xi(\omega))) \leq \alpha(\omega) d(\xi(\omega), T(\omega, \xi(\omega)))$.

It further implies that $(1 - \alpha(\omega)) d(\xi(\omega), T(\omega, \xi(\omega))) \leq 0$ for each $\omega \in \Omega$.

Therefore for all $\omega \in \Omega$,

$$d(\xi(\omega), T(\omega, \xi(\omega))) = 0 \text{ and hence } \xi(\omega) \in T(\omega, \xi(\omega)).$$

Similarly $\quad d(\xi(\omega), S(\omega, \xi(\omega))) \leq H(A(\omega), S(\omega, \xi(\omega)))$

$$= \lim_{n\to\infty} H(T(\omega, \xi_n(\omega)), S(\omega, \xi(\omega))).$$

Hence $\xi(\omega) \in S(\omega, \xi(\omega))$ for all $\omega \in \Omega$. Now for any $\omega \in \Omega$,

$$H(A(\omega), T(\omega, \xi(\omega))) = \lim_{n\to\infty} H(S(\omega, \xi_n(\omega)), T(\omega, \xi(\omega)))$$

$$d(\xi(\omega), T(\omega, \xi(\omega))) \leq \alpha(\omega) d(\xi(\omega), T(\omega, \xi(\omega))).$$

It further implies that $d(\xi(\omega), T(\omega, \xi(\omega))) = 0$.

It follows that $T(\omega, \xi(\omega)) = A(\omega) = \lim_{n\to\infty} T(\omega, \xi_n(\omega))$ for each $\omega \in \Omega$. Similarly

$$H(A(\omega), S(\omega, \xi(\omega))) = \lim_{n\to\infty} H(T(\omega, \xi_n(\omega)), S(\omega, \xi(\omega)))$$

$$d(\xi(\omega), S(\omega, \xi(\omega))) \leq \alpha(\omega), d(\xi(\omega), S(\omega, \xi(\omega))).$$

It further implies that $d(\xi(\omega), S(\omega, \xi(\omega))) = 0$.

It follows that $S(\omega, \xi(\omega)) = A(\omega) = \lim_{n\to\infty} S(\omega, \xi_n(\omega))$ for each $\omega \in \Omega$.

Acknowledgment

The authors thank the referees for their appreciation, valuable comments and suggestions.

Reference

1. Adomain, G., Theory of random system, Trans. 4th Prague Conf. Information theory Statist. *Decision Functions and Random Processes*, 205-222, 1967 (1965).
2. Bharuch-Reid, On the theory of random equations, *Proc. Symp. Appl. Math.*, **16th**, 1963, 40-69 (1963), *Amer. Math. Soc, Providence, Rhode Island*, (1964).
3. Beg, I., Jahanjir, A. and Azam, A., Random coincidence and fixed points for weakly compatible mappings in convex metric spaces, *Asian-European J. Math.,* **2(2)**,171-182 (2009).
4. Beg, I. and Shahzad, N., *Random fixed points of random multivalued operators on polish spaces nonlinear Anal.,* **20**, 835-847 (1993).
5. Chouhan, S.S., Common fixed point theorem for four random operator in Hilber space, *Adv. Appl. Sci. Res*., **4(2)**, 219-222 (2013).
6. Dhage, B.C., Bedre, S.V., Jadhav, N.S. and Kang, S.M., Common random fixed point theorems for a pair of generalized weakly contractive mappings in polish spaces, *Adv. Fixed Point Theory*, **4**, No. **1**, 12-24 (2014).
7. Fan, Ky, Extensions of two fixed point theorems of F.E. Browder, *Math. Z.*, **112**, 234-240 (1969).
8. Itoh, S., A random fixed point theorem for a multivalued contraction mapping, *Pacific J. Math*., **68**, 85-90 (1977).
9. Mehta, S. and Dhagat, V.B., Some fixed point theorems in polish spaces, *App. Math. Sci*., Vol. **4**, No. **28**, 1395-1403 (2010).
10. Nashine, H.K., New random fixed point results for generalized altering distance functions, *Sarajevo J. of Math.*, Vol. **7(20)**, 245-253 (2011).
11. Sanodia, P.L., Jaiswal, D. and Rajput, S.S., Some rational contractions in random spaces, *Int. J. of Adv. Sci. and Tec. Res*., Issue **2**,Vol. **1**, 294-304 (2012).
12. Shrivastava, R., Agarwal, S., Bhardwaj, R. and Yadava, R.N., Random fixed point theorems in polish spaces, *Int. J. of The. and Appl. Sci*., **4(1)**,16-19 (2012).

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