Acta Ciencia Indica, Vol. XL M, No. 2, 223 (2014)

A NOTE ON COMMON FIXED POINTS FOR MULTIVALUED MAPS

SHOYEB ALI SAYYED

Principal, Royal College of Technology, Indore (M.P.), India

LATA VYAS

Asst. Prof., Deptt. of Mathematics, Lakshmi Narain College of Technology, Indore (M.P.), India

AND

FARKHUNDA SAYYED

Professor, Deptt. of Mathematics, Lord Krishna College of Technology, Indore (M.P.), India

RECEIVED : 22 January, 2014

REVISED : 28 March, 2014

In this paper we have extended the result of Sayyed[10]. The purpose of this paper to further demonstrate the effectiveness of the compatible map concept as a mean of multivalued and single valued maps satisfying a contractive type condition.

KEYWORDS AND PHRASES: Hausdorff metric, Multivalued mappings, compatible mapping, complete metric space and coincidence point.

AMS (2010) subject classifications: primary 54H25; Secondary 47H10

INTRODUCTION

Banach obtained a fixed point theorem for contraction mapping, appearance of the celebrated Banach contraction principle, several generalizations of this theorem in the setting of point mappings have been obtained. Nadler [7] was the first to extend Banach contraction principle to multivalued contracting mapping.

Rhoades [9] gave a complete and comparison of various definitions of contraction mapping and also survey of the subject. The result is a generalized concept of commuting and compatible mappings under some conditions and corresponding result of Beg and Azam [1], Jungck [3, 4], Kaneko [5] Nadler [7], Reich [8], Sayyed, *et.al* [11] and many others. In this direction Faset, *et. al.* [2] proved a fixed point theory for multivalued generalized non expansive mappings. Recently Lateef *et al*. [6], Yadad *et al*. [13] and Wang and Song [12] proved a fixed point theorem for multivalued maps.

Preliminaries

Let (X, d) be a metric space and let $CB(X)$ denote the family of all non-empty bounded closed subsets of X. For $A, B \in CB(X)$, let $H(A, B)$ denote the distance between A and B in Hausdorff metric, that is

$$H(A, B) = \inf E_{AB}$$

where $$E_{AB} = \{\varepsilon > 0 : A \subset N(\varepsilon, B), B \subset N(\varepsilon, A)\}$$

$$N(\varepsilon, A) = \{x : d(x, A) < \varepsilon\}.$$

A point x is said to be a fixed point of a single valued mapping $f : X \to X$ (multivalued mapping $T : X \to CB(X)$) provided $x = fx$ ($x \in Tx$). The point x is called coincidence point of f and T, if $fx \in Tx$. If each element of X is a coincidence point of f and T, then f is called a selection of T.

Let $T : X \to CB(X)$ be a mapping, then $C_T = \{f : X \to X : TX \subset fX$ and $(\forall x \in X)$ $(f\,Tx = T\,fx)\}$. T and f are said to be commuting mappings if for each $x \in X$, $f(Tx) = fTx = Tfx = T(fx)$.

Lemma 2.1: {Beg [1, Lemma 2.1]}. Let S, T be two multivalued mappings of X into $CB(X)$. Let $x_0, x_1 \in X$. Then for each $y \in T(x_1)$ one has $d(y, Sx_0) \le H(Tx_1, Sx_0)$.

Theorem 2.2: Let S, T be two mappings from a complete metric space X into $CB(X)$ and let $f \in C_S \cap C_T$ be continuous mapping. Suppose that for all $x, y \in X$,

$$[H(Sx, Ty)]^2 \le \alpha[d(fx, Sx)d(fy, Ty) + d(fx, Ty)d(fy, Sx)]$$
$$+\beta[d(fx, Sx)\,d(fy, Sx) + d(fy, Ty)\,d(fx, Ty)]$$
$$+\gamma d(fx, fy)H(Sx, Ty) \qquad \text{....(1)}$$

where $\alpha, \beta, \gamma \ge 0$ and $0 \le \alpha + 2\beta + \gamma < 1$. Then there exists a common coincidence point of f and T and f and S.

Proof: Define $M = \dfrac{\alpha+\beta+\gamma}{1-\beta}$. Let x_0 be an arbitrary, but fixed element of X. We shall construct two sequences $\{x_n\}$ and $\{y_n\}$ as follows.

Let $x_1 \in X$ be such that $y_1 = fx_1 \in Sx_0$, using the definition of Hausdorff metric and fact that $Tx \subset fx$, we may choose $x_2 \in X$ such that $y_2 = fx_2 \in Tx_1$ and $d(y_1, y_2) = d(fx_1, fx_2) \le H(Sx_0, Tx_1) + (\alpha + \beta + \gamma)$.

Since $S(X) \subset f(x)$, we may choose $x_3 \in X$ such that $y_3 = fx_3 \in Sx_2$ and $d(y_2, y_3) = d(fx_2, fx_3) \le H(Tx_1, Sx_2) + \dfrac{(\alpha+\beta+\gamma)^2}{1-\beta}$.

By induction, we produce two sequence of points of X such that

$$y_{2k+1} = fx_{2k+1} \in Sx_{2k},$$
$$y_{2k+2} = fx_{2k+2} \in Tx_{2k+1}, \qquad \text{.....(2)}$$

where k is any positive integer. Further more

$$d(y_{2k+1}, y_{2k+2}) = d(fx_{2k+1}, fx_{2k+2})$$

$$\leq H(Sx_{2k}, Tx_{2k+1}) + \frac{(\alpha+\beta+\gamma)^{2k+1}}{(1-\beta)^{2k}}$$

$$d(y_{2k+2}, y_{2k+3}) = d(fx_{2k+2}, fx_{2k+3})$$

$$\leq H(Tx_{2k+1}, Sx_{2k+2}) + \frac{(\alpha+\beta+\gamma)^{2k+2}}{(1-\beta)^{2k+1}}$$

Hence

$$[d(fx_{2k+1}, fx_{2k+2})]^2 < \alpha[d(fx_{2k}, Sx_{2k})]\, d(fx_{2k+1}, Tx_{2k+1})$$

$$+ d(fx_{2k}, Tx_{2k+1})\, d(fx_{2k+1}, Sx_{2k})]$$

$$+\beta[d(fx_{2k}, Sx_{2k})\, d(fx_{2k+1}, Sx_{2k})$$

$$+ d(fx_{2k+1}, Tx_{2k+1})\, d(fx_{2k}, Tx_{2k+1})]$$

$$+\gamma d(fx_{2k}, fx_{2k+1})\, d(fx_{2k+1}, fx_{2k+2}) + \frac{(\alpha+\beta+\gamma)^{2k+1}}{(1-\beta)^{2k}}$$

$$d(fx_{2k+1}, fx_{2k+2}) < (\alpha+\beta+\gamma)\, d(fx_{2k}, fx_{2k+1}) + \beta d(fx_{2k+1}, fx_{2k+2})]$$

$$+ \frac{(\alpha+\beta+\gamma)^{2k+1}}{(1-\beta)^{2k}}$$

$$d(fx_{2k+1}, fx_{2k+2}) \leq \frac{(\alpha+\beta+\gamma)}{(1-\beta)} d(fx_{2k}, fx_{2k+1}) + \frac{(\alpha+\beta+\gamma)^{2k+1}}{(1-\beta)^{2k+1}}$$

$$d(fx_{2k+1}, fx_{2k+2}) \leq Md(fx_{2k}, fx_{2k+1}) + M^{2k+1}$$

Similarly, $d(fx_{2k}, fx_{2k+1}) \leq Hd(Tx_{2k}, Sx_{2k}) + \dfrac{(\alpha+\beta+\gamma)^{2k}}{(1-\beta)^{2k-1}}$

Therefore, $d(fx_{2k}, fx_{2k+1}) \leq Md(fx_{2k-1}, fx_{2k}) + M^{2k}$

It further implies that

$$d(y_n, y_{n+1}) \leq Md(y_{n-1}, y_n) + M^n$$

$$\leq M^{n-1} d(y_1, y_2) + (n-1)M^n$$

$$\leq M^{n-1} d(fx_1, fx_2) + (n-1)M^n$$

for $p \geq 1$, we have

$$d(y_{n+1}, y_{n+p+1}) \leq d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + ... + d(y_{n+p}, y_{n+p+1})$$

$$\leq \{M^n d(fx_1, fx_2) + nM^{n+1}\}$$

$$+\{M^{n+1}d(fx_1, fx_2)+(n+1)M^{n+2}\}+....$$

$$+\{M^{n+p-1}d(fx_1, fx_2)+(n+p-1)M^{n+p}\}$$

$$\leq \sum_{i=1}^{n+p-1} M^i d(fx_1, fx_2)+\sum_{i=n}^{n+p-1} iM^{i+1}$$

It follows that the sequence $\{y_n\}$ is Cauchy sequence. Hence there exists z in X such that $y_n \to z$. Therefore $fx_{2k+1} \to z$ and $fx_{2k+2} \to z$. From (2), we have

$$f^2x_{2k+1} = ffx_{2k+1} \in fSx_{2k} \subset Sfx_{2k},$$

and
$$f^2x_{2k+2} = ffx_{2k+2} \in fTx_{2k+1} \subset Tfx_{2k+1}..$$

Now using lemma 2.1

$$[d(fz, Sz)]^2 \leq [d(fz, f^2x_{2k+2})+d(f^2x_{2k+2}, Sz)]^2$$

$$\leq [d(fz, f^2x_{2k+2})+H(Tfx_{2k+1}, Sz)]^2$$

$$= [d(fz, f^2x_{2k+2})]^2 + 2H(Tfx_{2k+1}, Sz)d(fz, f^2x_{2k+2})$$

$$+[H(Tfx_{2k+1}, Sz)]^2$$

$$\leq [d(fz, f^2x_{2k+2})]^2 + 2H(Tfx_{2k+1}, Sz)d(fz, f^2x_{2k+2})$$

$$+ \alpha[d(fz, Sz)d(f^2x_{2k+1}, Tfx_{2k+1})+d(fz, Tfx_{2k+1})$$

$$d(f^2x_{2k+1}, Sz)] + \beta[d(fz, Sz)\,d(f^2x_{2k+1}, Sz)$$

$$+d(f^2x_{2k+1}, Tfx_{2k+1})\,d(fz, Tfx_{2k+1})]$$

$$+\upsilon d(fz, f^2x_{2k+1})\,H(Tfx_{2k+1}, Sz)$$

$$\leq [d(fz, f^2x_{2k+2})]^2 + 2H(Tfx_{2k+1}, Sz)d(fz, f^2x_{2k+2})$$

$$+\alpha[d(fz, Sz)d(f^2x_{2k+1}, f^2x_{2k+2})+d(fz, f^2x_{2k+2})$$

$$d(f^2x_{2k+1}, Sz)] + \beta[d(fz, Sz)\,d(f^2x_{2k+1}, Sz)$$

$$+ d(f^2x_{2k+1}, f^2x_{2k+2})d(fz, f^2x_{2k+2})]$$

$$+ \gamma d(fz, f^2x_{2k+1})\,d(f^2x_{2k+2}, Sz)$$

Since f is continuous, by letting $K \to \infty$, we obtain

$$[d(fz, Sz)]^2 \leq \beta[d(fz, Sz)]^2$$

or
$$d(fz, Sz) \leq \sqrt{\beta}d(fz, Sz).$$

Thus $fz \in Sz$. Similarly,

$$[d(fz,Tz)]^2 \le [d(fz,f^2x_{2k+1})+d(f^2x_{2k+1},Tz)]^2$$
$$\le [d(fz,f^2x_{2k+1})+H(Sfx_{2k},Tz)]^2$$
$$\le \beta[d(fz,Sz)]^2$$

Therefore $fz \in Tz$. Hence Z is a coincidence point of f and S and f and T.

Corollary 2.3: Let S, T be continuous mappings from a complete metric space X into $CB(X)$ and $f \in C_S \cap C_T$ be a continuous mapping. Assume that (1) is satisfied. If $f(z) \in Sz \cap Tz$ implies $\lim_{n\to\infty} f^n z = t$, then t is a common fixed point of S, T and f.

Proof: Clearly, $fx \in Sz$ implies that $f^2 z \in fSz \subset Sfz$. Therefore $f^{n+1}z \in Sf^n Z$. If follows that $t \in St$. Similarly $t \in Tt$. Moreover.

$$ft = f \lim_{n\to\infty} f^n z = \lim_{n\to\infty} f^{n+1} z = t..$$

Hence t is a common fixed point of f, S and T.

In the following theorem the continuity of f and its commutativity with S and T are not required.

Theorem 2.4: Let S, T be two mappings from a metric space X into $CB(X)$ and let $f: X \to X$ be a mapping such that $f(X)$ is complete, $T(X) \subset f(X)$ and $S(X) \subset f(X)$. Suppose that (1) is satisfied, then there exists a common coincidence point of f and T and f and S.

Proof: As in the proof of theorem 2.2 we construct the Cauchy sequence $y_n = fx_n \in X$. By our hypothesis it follows that there exists a point u in X such that $y_n \to z = fu$. Now using Lemma 2.1, we have

$$[d(fu,Tu)]^2 \le [d(fu,fx_{2k+1})+d(fx_{2k+1},Tu)]^2$$
$$\le [d(fu,fx_{2k+1})+H(Sx_{2k},Tu)]^2$$
$$\le [d(fu,fx_{2k+1})]^2 + 2H(Sx_{2k},Tu)d(fu,fx_{2k+1})$$
$$+[H(Sx_{2k},Tu)]^2$$
$$\le [d(fu,fx_{2k+1})]^2 + 2d(fu,fx_{2k+1})H(Sx_{2k},Tu)$$
$$+\alpha[d(fx_{2k},Sx_{2k})d(fu,Tu)+d(fx_{2k},Tu)\ d(fu,Sx_{2k})]$$
$$+\beta[d(fx_{2k},Sx_{2k})d(fu,Sx_{2k})+d(fu,Tu)$$
$$d(fx_{2k},Tu)]+\gamma d(fx_{2k},fu)H(Sx_{2k},Tu)$$
$$\le [d(fu,fx_{2k+1})]^2 + 2d(fu,fx_{2k+1})d(fx_{2k+1},Tu)$$
$$+\alpha[d(fx_{2k},fx_{2k+1})d(fu,Tu)+d(fx_{2k},Tu)\ d(fu,fx_{2k+1})]$$
$$+\beta[d(fx_{2k},fx_{2k+1})d(fu,fx_{2k+1})+d(fu,Tu)$$
$$d(fx_{2k},Tu)]+\upsilon d(fx_{2k},fu)d(fx_{2k+1},Tu).$$

Letting $k \to \infty$, we obtain

$$[d(fu,Tu)]^2 \leq \beta[d(fu,Tu)]^2$$

or

$$d(fu,Tu) \leq \sqrt{\beta} d(fu,Tu)$$

Hence $fu \in Tu$. Similarly,

$$[d(fu,Su)]^2 \leq [d(fu,fx_{2k+2}) + d(fx_{2k+2},Su)]^2$$

$$\leq d(fu,fx_{2k+2}) + H(Tx_{2k+1},Su)]^2$$

$$\leq \beta[d(fu,Su)]^2$$

Hence $fu \in Su$.

Reference

1. Beg, I. and Azam, A., Common fixed points for commuting and compatible maps, *Discussiones Mathemticae Differential Inclusions*, **16**, 121-135 (1996).
2. Falset, J.G., Fuster, E.L. and Galvez, E.M., Fixed point theory for multivalued generalized non expansive mappings, *Appl. Anal. Discrete Math*., **6**, 265-286 (2012).
3. Jungck, G., Commuting mapping and fixed points, *Amer. Math. Monthly*, **83**, 261-263 (1976).
4. Jungck, G., Common fixed points for commuting and compatible maps on compacta, *Proc. Amer. Math. Soc*., **103(3)**, 977-983 (1988).
5. Kaneko, H., Single valued and multivalued *f*-contraction, *Boll. U.M.I*., **4A**, 29-33 (1985).
6. Lateef, D., Sayyed, S.A. and Bhattacharyya, A., Commom fixed point for multivalued and compatible maps, *Ultra Scientist*, Vol. **21(2)**M, 503-508 (2009).
7. Nadler, S.B., Jr. Multivalued contraction mappings, *Pacific J. Math*., **30**, 475-488 (1969).
8. Reich, S., Some remarks concerning contraction mappings, *Canad Math. Bull*., **14**, 121-124 (1971).
9. Rhoades, B.E., A comparison of various definitions of contractive mappings, *Trans, Amer. Math. Soc.*, **226**, 257-290 (1977).
10. Sayyed, S.A.,Some results on common fixed point for multivalued and compatible maps, *Ultra Engineer*, Vol. **1(2)**, 191-194 (2012).
11. Sayyed, S.A., Sayyed, F. and Badshah, V.H., Fixed point theorem and multivalued mappings, *Acta Ciencia Indica*, Vol. **XXXVIII** M, No. **2**, 155-158 (2002).
12. Wang, Q., Song, M., Commom fixed point theorems of multivalued maps in ultrametric spaces, *Applied Mathematics*, **4**, 17-420 (2013).
13. Yadav, H., Sayyed, S.A. and Babshah, V.H., Fixed point theorem for multivalued mappings satisfying functional inequality, *Oriental Journal of Computer Science and Technology*, Vol. **4(1),** 221-223 (2011).

❑