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ON THE SYSTEM OF DOUBLE EQUATIONS $b - T = x^2,\ \frac{b}{2} - T = y^2,\ T \neq a$ SQUARE

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The number 10 has the peculiar property that if unity is subtracted to it, the difference is a perfect square, 9 and if unity is subtracted to its half, 5, the result, 4, is also a perfect square. There are infinity of numbers satisfying the above pattern and they are obtained by solving the system of equations $b+1=x^2$ and $\frac{b}{2}+1=y^2$ and this result has appeared in [1]. This property has motivated us to search for non-zero integers b and T ($\neq$ a square) such that $b-T=x^2$, $\frac{b}{2}-T=y^2$. The system of double equations given by $b-T=x^2, \frac{b}{2}-T=y^2, T \neq a$ square is analysed for its non-trivial integral solutions. A few interesting properties are also presented and discussed.

KEYWORDS : The double equations, the recurrence relations satisfied by the solutions, the few interesting properties.

2010 Mathematics Subject Classifications: 11A, 11D

INTRODUCTION

The number10 has the peculiar property that if unity is subtracted to it, the difference is a perfect square, 9 and if unity is subtracted to its half, 5, the result, 4, is also a perfect square. There are infinity of numbers satisfying the above pattern and they are obtained by solving the system of equations $b+1=x^2$ and $\frac{b}{2}+1=y^2$ and this result has appeared in [1]. This property has motivated us to search for non-zero integers b and T ($\neq$ a square) such that $b-T=x^2$, $\frac{b}{2}-T=y^2$ The recurrence relations satisfied by the solutions are also given. In

[3], we have analysed the system of double equations $b+T=x^2, \frac{b}{2}+T=y^2, T \neq a$ square for its non-trivial integral solutions and a few interesting properties have presented.

Method of analysis

Let b and T ($\neq$ a square) be any two non-zero integers such that

$$b-T=x^2 \quad \ldots (1.1)$$

$$\frac{b}{2}-T=y^2 \quad \ldots (1.2)$$

Eliminating b, we get the Pell equation

$$x^2-2y^2=T. \quad \ldots (1.3)$$

The general solutions [2] of (1.1), (1.2) are given by

$$x_n=\left\{\frac{1}{2}[(3+2\sqrt{2})^{n+1}(x_0+\sqrt{2}y_0)+(3-2\sqrt{2})^{n+1}(x_0-\sqrt{2}y_0)]\right\}, \quad \ldots (1.4)$$

$$y_n=\left\{\frac{1}{2\sqrt{2}}[(3+2\sqrt{2})^{n+1}(x_0+\sqrt{2}y_0)-(3-2\sqrt{2})^{n+1}(x_0-\sqrt{2}y_0)]\right\}, \quad \ldots (1.5)$$

where $(x_0+\sqrt{2}\,y_0)$ is the fundamental solution of (1.3).

Thus, knowing the values of x_n, y_n in (1.1, 1.2), the sequence of values of b are obtained. In particular from (1.1) and (1.4), we get

$$b=T+\left\{\frac{1}{2}[(3+2\sqrt{2})^{n+1}(x_0+\sqrt{2}y_0)+(3-2\sqrt{2})^{n+1}(x_0-\sqrt{2}y_0)]\right\}^2, \quad \ldots (1.6)$$

$n = 0, 1, 2, 3,\ldots..$

Case (I): When $T=\alpha^2+2\alpha-1$, the equation (1.6) becomes

$$b=\alpha^2+2\alpha-1+\left\{\frac{1}{2}[(3+2\sqrt{2})^{n+1}(\alpha+1+\sqrt{2})+(3-2\sqrt{2})^{n+1}(\alpha+1-\sqrt{2})]\right\}^2, \quad \ldots (1.7)$$

where $x_0+\sqrt{2}\,y_0=\alpha+1+\sqrt{2}$ is the fundamental solution of $x^2-2y^2=\alpha^2+2\alpha-1$.

For the sake of simplicity a few solutions of (1.7) for T = 2, 7, 14 are presented in the table 1.

Table 1

Serial No.	The values of n	The solutions b for $T = 2$	The solutions b for $T = 7$	The Solutions b for $T = 14$
1	0	102	176	270
2	1	3366	5632	8478
3	2	114246	190976	287310
4	3	3083902	6487216	9759390

5	4	131836326	220374032	331531278
6	5	4478554086	7486229536	11262303390

It is interesting to note that all the solutions obtained in this case are even. When $T = 2, 7, 14$, the solutions of last digits form the following patterns respectively:

2 6 6; 6 2 6; 0 8 0

as seen from the above table.

Further the solutions satisfy the following recurrence relation:

(a) Recurrence relations for solution (b_α) among different values of T:

(i) $[b_{\alpha+2} - c(\alpha+2)]^{1/2} - 2[b_{\alpha+1} - c(\alpha+1)]^{1/2} + [b_\alpha - c(\alpha)]^{1/2} = 0$

where $c(\alpha) = \alpha^2 + 2\alpha - 1$.

In particular for $c(3) = 14$, $c(2) = 7$ and $c(1) = 2$, when $\alpha = 1$, we have

(ii) $[b_3 - 14]^{1/2} - 2[b_2 - 7]^{1/2} + [b_1 - 2]^{1/2} = 0,$ when $n = 0$.

(b) Recurrence relations for solutions (b_n^α) among the particular value of T:

(i) $\left[b_{n+2}^{(\alpha)} - C\right]^{1/2} - 6\left[b_{n+1}^{(\alpha)} - C\right]^{1/2} + \left[b_n^{(\alpha)} - C\right]^{1/2} = 0,$ where $C = \alpha^2 + 2\alpha - 1$.

In particular for $C = 2$ when $\alpha = 1$ and $C = 7$ when $\alpha = 2$, We have

(ii) $\left[b_3^{(1)} - 2\right]^{1/2} - 6\left[b_2^{(1)} - 2\right]^{1/2} + \left[b_1^{(1)} - 2\right]^{1/2} = 0$, when $n = 1$.

(iii) $\left[b_3^{(2)} - 7\right]^{1/2} - 6\left[b_2^{(2)} - 7\right]^{1/2} + \left[b_1^{(2)} - 7\right]^{1/2} = 0,$ when $n = 1$

Case (ii) : When $T = \alpha^2 + 4\alpha - 4$, the equation (1.6) becomes

$$b = \alpha^2 + 4\alpha - 4 + \left\{\frac{1}{2}[(3+2\sqrt{2})^{n+1}(\alpha+2+2\sqrt{2}) + (3-2\sqrt{2})^{n+1}(\alpha+2-2\sqrt{2})]\right\}^2, \quad \ldots (1.8)$$

where $x_0 + \sqrt{2}y_0 = (\alpha+2) + 2\sqrt{2}y_0$ is the fundamental solution of

$$x^2 - 2y^2 = \alpha^2 + 4\alpha - 4. \quad \ldots (1.9)$$

For the sake of simplicity a few solutions of (1.8) for $T = 1, 8, 17$ are presented in the table 2.

Table 2

Serial No.	The values of *n*	The solutions b for *T* = 1	The solutions b for *T* = 8	The Solutions b for *T* = 17
1	0	10	24	42
2	1	290	408	546
3	2	9802	13464	17706
4	3	332930	456984	600642

5	4	11309770	14899608	20403306
6	5	384199202	527345304	693110946

It is interesting to note that all the solutions obtained in this case are even. When $T = 1, 8, 17$, the solutions of last digits form the following patterns:

0 0 2; 4 8 4; 2 6 6

as seen from the above table.

Further the solutions satisfy the following recurrence relations:

(a) Recurrence relations for solutions (b_α) among different values of T:

(i) $[b_{\alpha+2} - c(\alpha+2)]^{1/2} - 2[b_{\alpha+1} - c(\alpha+1)]^{1/2} + [b_\alpha - c(\alpha)]^{1/2} = 0$

where $c(\alpha) = \alpha^2 + 4\alpha - 4$.

In particular for $c(3) = 17$, $c(2) = 8$ and $c(1) = 1$, when $\alpha = 1$, we have

(ii) $[b_3 - 14)]^{1/2} - 2[b_2 - 7]^{1/2} + [b_1 - 2]^{1/2} = 0,$ when $n = 0$.

(b) Recurrence relations for solutions (b_n^α) among the particular value of T:

(i) $\left[b_{n+2}^{(\alpha)} - C\right]^{1/2} - 6\left[b_{n+1}^{(\alpha)} - C\right]^{1/2} + \left[b_n^{(\alpha)} - C\right]^{1/2} = 0,$ where $C = \alpha^2 + 4\alpha - 4$.

In particular for $C = 1$ when $\alpha = 1$ and $C = 8$ when $\alpha = 2$, we have

(ii) $\left[b_3^{(1)} - 1\right]^{1/2} - 6\left[b_2^{(1)} - 1\right]^{1/2} + \left[b_1^{(1)} - 1\right]^{1/2} = 0,$ when $n = 1$.

(iii) $\left[b_3^{(2)} - 8\right]^{1/2} - 6\left[b_2^{(2)} - 8\right]^{1/2} + \left[b_1^{(2)} - 8\right]^{1/2} = 0,$ when $n = 1$.

Conclusion

In this paper, we have analysed for its non-trivial integral solutions of the system of double equations: $b - T = x^2, \frac{b}{2} - T = y^2, T \neq a$ square and a few interesting properties have presented. To conclude one may search for other non-trivial integral solutions of the system of the above double equations.

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