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ON CERTAIN GENERATING FUNCTIONS OF BIORTHOGONAL POLYNOMIALS SUGGESTED BY LAGUERRE POLYNOMIALS

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In this note we obtain a novel theorem on the extension of bilateral generating functions involving biorthogonal polynomials suggested by Laguerre polynomials derived by Shreshtha and Bajracharya from the group theoretic view point. At first we introduce a novel linear partial differential operator and the extended group corresponding to the operator and finally, we obtain our desired result by applying the operator and group effect on a unilateral generating function involving the polynomials under consideration. Some application of our result is also given here.

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INTRODUCTION

In [1, 2], J.D.E. Konhauser discussed the biorthogonality and some other properties of $Y_n^{\alpha}(x;k)$ and $Z_n^{\alpha}(x;k)$ for any positive integer k, where $Y_n^{\alpha}(x;k)$ is a polynomial in x and $Z_n^{\alpha}(x;k)$ is a polynomial in $x^k, \alpha \rangle -1, k$ is a positive integer. For $k=1,$ these polynomials reduce to the generalized Laguerre polynomials $L_n^{\alpha}(x)$ [3]. In the present paper we are interested only on $Y_n^{\alpha}(x;k)$. An explicit representation for the polynomials $Y_n^{\alpha}(x;k)$ was given by Carlitz [4] in the following form:

$$Y_n^{\alpha}(x;k) = \frac{1}{n!}\sum_{i=0}^{n}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_n,$$

where $(a)_n$ is a Pochhammer symbol [5].

In [6], Shreshtha and Bajracharya obtain the following theorem on bilateral generating function involving $Y_n^{\alpha}(x;k)$ by using one parameter group of continuous transformations.

Theorem-1 : If

$$G(x,w)=\sum_{n=0}^{\infty} a_n\ Y_n^{\alpha}(x;k)\ w^n$$

then

$$\left(1-kt\right)^{-\frac{\alpha+1}{k}} \exp\left[x\left\{1-(1-kt)^{-\frac{1}{k}}\right\}\right] G\left(x(1-kt)^{-\frac{1}{k}},\ yt(1-kt)^{-1}\right).$$
$$=\sum_{n=0}^{\infty} f_n(y)\ Y_n^{\alpha}(x;k)\ t^n$$

where $$f_n(y)=\sum_{p=0}^{n} a_p\ k^{n-p}\ \binom{n}{p}\ y^p\ .$$

The aim at writing this note is to prove the following theorem on the extension of the above theorem on bilateral generating functions involving biorthogonal polynomials suggested by Laguerre polynomials with the help of group-theoretic method.

Theorem-2: If

$$G(x,w)=\sum_{n=0}^{\infty} a_n\ Y_{n+m}^{\alpha}(x;k)\ w^n \qquad \ldots (1.2)$$

then

$$\left(1-kt\right)^{-m-\frac{\alpha+1}{k}} \exp\left[x\left\{1-(1-kt)^{-\frac{1}{k}}\right\}\right]\times G\left(x(1-kt)^{-\frac{1}{k}},\ yt(1-kt)^{-1}\right).$$

$$=\sum_{n=0}^{\infty} f_n(y)\ \ Y_{n+m}^{\alpha}(x;k)\ \ t^n \qquad \ldots (1.3)$$

where $$f_n(y)=\sum_{p=0}^{n} a_p\ k^{n-p}\ \binom{n+m}{p+m}\ y^p\ ,$$

which does not seem to have appeared before.

The importance of the above theorem lies in the fact that whenever one knows a unilateral generating relation of type (1.2), then the corresponding bilateral generating relation can at once be written down from (1.3). Thus a large number of bilateral generating relations can be obtained by attributing different suitable values to a_n in (1.2).

Proof of the Theorem-2

For the polynomials $Y_n^{\alpha}(x;k)$, we first define the following linear partial differential operator R:

$$R=xz\frac{\partial}{\partial x}+kz^2\frac{\partial}{\partial z}\ +z(\ km+\alpha-x+1)$$

such that

$$R\ (Y_{n+m}^{\alpha}(x\ ;k)\ z^n)=k(n+m+1)Y_{n+m+1}^{\alpha}(x\ ;k)\ z^{n+1}. \qquad \ldots (2.1)$$

The extended form of the group generated by R is given by,

$$\exp(wR)\ f(x,z) = \left(1-kwz\right)^{-m-\frac{\alpha+1}{k}}\ \exp\left[x\left\{1-\frac{1}{(1-kwz)^{\frac{1}{k}}}\right\}\right]\times f\left(\frac{x}{(1-kwz)^{\frac{1}{k}}},\frac{z}{(1-kwz)}\right). \qquad \ldots (2.2)$$

Let us consider the following generating relation,

$$G\left(x,w\right) = \sum_{n=0}^{\infty} a_n\ Y_{n+m}^{\alpha}(x;k)\ w^n . \qquad \ldots (2.3)$$

Replacing w by wyz and then operating $\exp(wR)$ on both sides, we get

$$\exp(wR)G\left(x,wyz\right) = \exp(wR)\sum_{n=0}^{\infty} a_n\ \left(\ Y_{n+m}^{\alpha}(x;k)\ z^n\right)\ \left(wy\right)^n . \qquad \ldots (2.4)$$

Using (2.2), then left hand side of (2.4) become

$$\left(1-kwz\right)^{-m-\frac{\alpha+1}{k}}\ \exp\left[x\left\{1-\frac{1}{(1-kwz)^{\frac{1}{k}}}\right\}\right]\times G\left(\frac{x}{(1-kwz)^{\frac{1}{k}}},\frac{wyz}{(1-kwz)}\right). \qquad \ldots (2.5)$$

Again using (2.1), then right hand side of (2.5) becomes

$$\sum_{n=0}^{\infty}\ \sum_{p=0}^{n} a_{n-p}\ k^{n-p}\ \binom{n+m}{p}\ Y_{n+m}^{\alpha}(x;k)\ (wz)^n y^{n-p} \qquad \ldots (2.6)$$

Now equating (2.5) and (2.6) and then replacing $wz = t$, we get

$$\left(1-kt\right)^{-m-\frac{\alpha+1}{k}}\ \exp\left[x\left\{\ 1-(1-kt)^{-\frac{1}{k}}\ \right\}\ \right]G\left(x(1-kt)^{-\frac{1}{k}},\ yt(1-kt)^{-1}\right).$$

$$= \sum_{n=0}^{\infty} f_n(y)\ \ Y_{n+m}^{\alpha}(x;k)\ \ t^n \qquad \ldots (2.7)$$

where $$f_n(y) = \sum_{p=0}^{n} a_p\ k^{n-p}\ \binom{n+m}{p+m} y^p .$$

This completes the proof of the Theorem-2.

Corollary 1: Now putting $m = 0$ in (2.7), we get the result found derived in [6].

We now discuss some special cases:

Special Case 1. If we put $k = 1$ we get the corresponding result involving generalized Laguerre polynomials found derived in [7] .

Result 2: Putting $m = 0$ in Special case 1, we get the result found derived in [8].

Application

Below we give an application of our main result (Theorem-2).

At first we consider the following generating relation [4]:

$$\left(1-u\right)^{\frac{-(\alpha+mk+1)}{k}} \exp\left[x\{1-(1-u)^{-\frac{1}{k}}\}\right]\times Y_n^{\alpha}\left(x(1-u)^{-\frac{1}{k}}\,;k\right)$$

$$= \sum_{n=0}^{\infty}\frac{(n+m)!}{n!\,m!}\,Y_{n+m}^{\alpha}(x\,;k)\,u^n \qquad \ldots (3.1)$$

If take $a_n = \dfrac{(n+m)!}{n!\,m!}$, we get

$$G(x,u) = \left(1-u\right)^{\frac{-(\alpha+mk+1)}{k}} \exp\left[x\{1-(1-u)^{-\frac{1}{k}}\}\right]\times Y_n^{\alpha}\left(x(1-u)^{-\frac{1}{k}}\,;k\right). \qquad \ldots (3.2)$$

By the application of our theorem we get the following generalization of the result (3.1) .

$$\left(1-kt-yt\right)^{-\frac{\alpha+mk+1}{k}} \exp\left[x\left\{1-(1-kt-yt)^{-\frac{1}{k}}\right\}\right] Y_n^{\alpha}\left(x(1-kt-yt)^{\frac{-1}{k}}\,;k\right)$$

$$= \sum_{n=0}^{\infty} f_n(y)\,Y_{n+m}^{\alpha}(x;u)\,w^n$$

where

$$f_n(y) = \sum_{p=0}^{\infty}\binom{p+m}{p}\binom{n+m}{p+m}y^p$$

Conclusion

From the above discussion, it may be concluded that any unilateral generating functions of type (1.2) may be immediately generalized with the help of the relation (1.3).

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