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EXISTENCE OF MEASURABLE SELECTOR EFFORS MEASURABLE, SCALARLY MEASURABLE SELECTOR OF MEASURABLE MULTIFUNCTIONS

DR. S.K. PANDEY

Deptt. of Mathematics, Govt. College Maihar, Satna (M.P.), India

AND

DR. R.S. PATEL

Deptt. of Mathematics, Govt. Auto. P.G. College, Satna (M.P.), India

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We study the existence of measurable selectors for multifunctions whose values are weekly compact subset of a Banach Space. One side we characterize multifunction having strongly measurable selector on the other side we prove that every scalary measurable multi function admits measurable selectors. We try to estabilish relation between measurable selector, effors measurable and scalarly measurable selector. Our work is extension of Cascales, Kodet & Rodrigues [2].

KEY WORDS: Multifunction measurable selector, Banach Space efforts measurable, Labesgue dominated convergence theorem.

Introduction

Let us suppose p be a projection from $Y \times X$ onto Y, $B \subset Y \times X$ aset and define $\Omega = p(B)$. A uniformization of B is a function $f : \Omega \to X$ such that $(t, f(t)) \in B$ for each $t \in \Omega$. Notice that with the aid of the axiom of choice such a uniformization f always exists. The problem is how nice can f be choosen when B is nice ? For instance if B is Borel measurable (Y and X are topological spaces) can f be chosen being Borel measurable? The study of the existence of nice uniformizations for Borel sets when $Y = X = \Omega = [0, 1]$ attracted the attention of leading mathematicians from the very begning of the XX century such as Baire, Borel, Hadamard, Labesgue, Von Neumann, Novikov, Kondo, Yankov, Luzin, Scerpinski etc. and precipitated the birth and flourishment of the descriptive set theory. More recent authors contributing to this topic are amongst others, Kuratowski, Ryll Nardzewski, Sion , Larman Mauldni, Pol, Saint-Raymond etc. Notice that for our given B we naturally can define the multifunction $F : \Omega \to 2^X$ that at each $t \in \Omega$ is given by $F(t) = \{x \in X : (t, x) \in B\}$ with this language properties of B are just properties of the graph of F defind as Graph $(F) = \{(t, x) : t \in \Omega, x \in F(t)\}$ and uniformization of B is just a selector of F *i.e.* a single valued function $f : \Omega \to X$ such that $f(t) \in F(t)$ for each $t \in \Omega$. When dealing with general multifunctions, the domain Ω is a usually a measurable or a topological space and the range X is usually a topological space. In this setting analysts, topologist and applied mathematicians soon realized that many times when one needs to find a nice selector f for F, the starting point is not a hypothesis about Graph (F).

B. Calcales, Kadets, V. and Rodrigues, J. [2, 3] Chistyakov, V.V. [4] Darinka Dentcheva [5] Dragan Djurcie, Ljubisa D.R. Kocinac [6] Pandey S.K. [10] are eminent authors in this field. Throughout in this paper (Ω, Σ, μ) is a complete finite measure space and X is a real Banach space. By 2^X we denote the family of all non-empty subset of X and by cl(X), K (X), *wk* (X) and *cwk* (X). We denote respectively the subfamilies of 2^X made up of norm closed, norm compact weakly compact and convex, weakly compact subsets of X.

A multifunction $F : \Omega \to 2^X$ that satisfies property (1.1) in following theorem Kuratowski- Ryll Nordzewski [9] is said to be Effors meaasurable. A single valued function $f : \Omega \to X$ is strongly measurable if it is the μ almost every where limit of sequence of Σ–simple X-valued functions defined in Ω Letus assume that X is separable and take a multifunction $F : \Omega \to \text{cl}(X)$ Effors measurable, then apply Kuratowski- Ryll Nordzewski's theorem to produce a Σ–Borel measurable selector f of F and then with the help of pettis measurability theorem f is strongly measurable.

Theorem (Kuratowski-Ryll Nordzewski's) : Let (Ω, Σ) be a measurable space and X a separable metric space. Let $F : \Omega \to 2^X$ be a multifunction with complete non-empty values satisfying that

$$\{t \in \Omega \ : F(t) \cap G \neq \phi\} \in \Sigma \qquad \ldots (1.1)$$

For each open set GcX then F admits a Σ–Borel(X) measurable selector f.

3. Notation, Definitions and Propositions : For the real Banach space $(X, \|, \|)$ we denote by B_X the close unit ball and S_X the unit sphere. For a set $D \subset X$ we define

$$\text{diam}(D) = \sup_{x,\, y \in D} \| x - y \|$$

and we denote by *co* (D) the convex hull of D. Given a multifunction $F : \Omega \to 2^X$ and $C \subset X$, we write

$$F^{-1}(C) = \{t \in \Omega : F(t) \cap c \neq \phi\}$$

Definition 3.1 : We says at a multifunction $F : \Omega \to 2^X$ satisfied property (p) if for each $\varepsilon > 0$ and each $A \in \Sigma^+$ there exists $B \in \Sigma^+{}_A$ and $D \subset X$ with diam $(D) \leq \varepsilon$ such that $F(t) \cap D \neq \phi$ for every $t \in B$. Let Σ^+ the family of all $A \in \Sigma$ with $\mu(A) > 0$, given $A \in \Sigma^+$ the collection of all subsets of A belonging to Σ^+ is denoted by Σ^+_A.

Proposition 3.2 : For a function $f : \Omega \to X$ the following statements are equivalent

(i) f satisfies property (P)

(ii) for each $e > 0$ and each S^+ there exists $B\hat{I}\, S^+_A$ with diam $(f(B)) \leq e$.

(iii) f is strongly measurable

Proposition 3.3 : Let $f : \Omega \to 2^x$ be a multifunction

(i) If there exists a multifunction $G : \Omega \to 2^x$ satisfying property (p) such that $G(t) \subset F(t)$ for μ *.a.e* $t \in \Omega$, then F satisfies property (P) as well.

(ii) If F admits strongly measurable selectors, then F satisfies property (p).

Proposition 3.4 : Suppose that X is separable. Let $F : \Omega \to \omega\kappa(X)$ be a multifunction.

The following statements are equivalent:

(i) F is effors measurable.

(ii) F^{-1} (ω) $\in \Sigma$ for every set $\omega \subset x$ which can be written as a finite intersection of closed half spaces.

Main result

Theorem 4.1 : Let us suppose $F :: \Omega \to \omega k\ (X)$ is measurable multifunctiom admits a strongly measurable selector, satisfies property (p), then threre exists a set of measure zero $\Omega_0 \in \Sigma$, a separable subspaces $Y \subset X$ and a multifunction $G : \Omega \backslash \Omega_0 \to \omega k\ (Y)$ that is Effors measurable and such that $G\ (t) \subset F\ (t)$ for every $t \in \Omega \backslash \Omega_0$.

Proof : In order to prove this, we took following steps :

Step 1: Let us combining property (p) and a standard exhaustion argument, we can find a countable partition (up to a μ -null set) $_1 = (A_n, 1)$ of Ω in Σ^+ and a sequence $(D_n, 1)$ of subset of X with diam $(D_n, 1) \leq \varepsilon_1$ such that $f\ (t) \cap D_n, 1 \neq \phi$ for every $t \in A_n$, 1 and every $n \in N$.

Observe that the set $V_{n,\,1,\,1} = \cap_{t \,\in\, An,\,1} (f\ (t) + \varepsilon_1\ Bx)$ contains D_n, 1 and so it is nonempty for every $n \in N$. The set $E_1 = \Omega \backslash Un \in NA_n, 1 \in \Sigma$ has measure zero. The same argument , but now with ε_2 instead of ε_1 allows us to find a countable partition $_2 = (A_n, 2)$ of Ω in Σ such that the $V_{n,\,2,\,2} = \cap_{t \,\in\, An,\,2}\ f\ (t)\ \varepsilon_2\ Bx)$ is the non empty for every $n \in N$ since $\varepsilon_1 \geq \varepsilon_2$ we also have $V_{n,\,2,\,1} = \cap_{t \,\in\, An,\,2}\ f\ (t)\ \varepsilon_1\ Bx) \neq \phi$ for every $n \in N$. Again the set $E_2 = \Omega \backslash Un \in NA_n,\ 2 \in \Sigma$ has measure zero.

In this we can find a sequence $\Gamma_m = (A_n, m)$ of countable partitions (upto a μ-null set E_m) of Ω in Σ^+ such that the sets $V_{n,\,m,\,k} = \cap_{t \,\in\, An,\,m}\ f\ (t)\ \varepsilon_\kappa\ Bx)$ for every $\kappa \leq m$ and every $n \in N$. clearly the set $\Omega_0 = U_{m \,\in\, N}\ E_m$ has measure zero.

Take $v_{n,\,m,\,k} \in V_{n,\,m,\,k}$ for every $k \leq m$ and every $n \in N$ and let Y be a closed linear space of all the $v_{n,\,m,\,k}\,s$ so that Y is separable. Since F has weakly compact values, it is clear that each $W_{n,\,m,\,k} = V_{n,\,m,\,k} \cap Y$ is weakly closed and non empty set.

Given $k \leq m$, set $F_{m,\,k} : \Omega \backslash \Omega_0 \to 2^Y$ by $F_{m,\,k} = \Sigma_{n \,\in\, N}\ W_{n,\,m,\,k} / A_{n,\,m}$ observe that for each set $C \subset Y$ we have $F^{-1}_{m,\,k}\ (C) \in \Sigma$.

Given $k \in N$, we define $F_K : \Omega \backslash \Omega_0 \to 2^Y$ by $F_K\ (t) = \text{clw}\ (\cup_{m \,\geq\, k}\ Fm,\ k\ (t))$ it is easy to see that for each weakly open set $U \subset Y$ we have $F^-_k\ (U) \in \Sigma$.

Step 2 : Fix $t \in \Omega \backslash \Omega_0$. For each $m \in N$, let $n_m\ (t) \in N$ be such that $t \in An_m\ (t)\ m$. Observe that for each $k \in N$ we have $F_k\ (t) \supset F_{k\,+\,1}\ (t)$ because the inequality $\in_{k\,+\,1} \leq \in_k$ allow us to write

$$\bigcup_{m \geq \kappa+1} F_{m,k+1}(t) = \bigcup_{m \geq \kappa+1} \ \bigcap_{s \in Anm(t),\ m} (F\ (s) + \varepsilon_{k+1} Bx) \bigcap Y$$

$$\subset \bigcup_{m \geq \kappa} \bigcap_{s \in Anm(t),\, m} (F(s) + \varepsilon_k Bx) \bigcap Y = \bigcup_{m \geq \kappa} Fm,\, k(t)$$

Set $G(t) = \bigcap_{k \in N} (F_k(t)$. We will prove that the weakly closed set $G(t)$ is non empty and contained in $F(t)$.

We have

$$G(t) \subset F_k(t) = \text{clw}\,(\bigcup_{m \geq \kappa} \bigcap_{s \in Anm(t),\, m} (F(s) + \varepsilon_k Bx) \cap Y) \subset F(t) + \varepsilon_k Bx \qquad ...(4.1.1)$$

for each $k \in N$, we take $x_k \in F_k(t)$ and write $x_k = y_k + z_k$ with $y_k \in F(t)$ and $z_k \in \varepsilon_k Bx$. Since the sequence $<y_k>$ is contained in the weakly compact set $F(t)$, it has a weak cluster point $y \in F(t)$. Since $z_k \to 0$ in norm as $k \to \propto$, we conclude that y is also a weak cluster point of (x_K).

Taking into account that $F_{k+1}(t) \subset F_k(t)$ for all $k \in N$.

It follows that $\quad y \in \bigcap_{k \in N} F_k(t) = G(t)$

$$\Rightarrow G(t) \subset F(t).$$

Step 3: It follows that G is a multifunction on $\Omega \backslash \Omega_0$ taking values in $wk(y)$.

Now we shall prove G is effors measurable.

In order to prove this we use proposition 3.4 let us fix $W \subset Y$ of the form

$w = \bigcap_{i=1}^{p} \{y \in Y : y_i^*(y) \leq a_i\}$ where $y_i^* \in Y^*$ and $a_i \in R$ for all $1 \leq i \leq p$. for each $k \in N$.

We define

$O_k = \bigcap_{i=1}^{p} \{y \in Y : y_i^*(y) < a_i + 1/k\}$ each O_k is weakly open in Y and so $F_k^-(O_k) \in \Sigma$.

We observe that $O_{k+1} \subset O_{k+1} \subset O_k$ for all $k \in N$ and that $W = \bigcap_{k \in N} O_k$. We claim that

$$G^{-1}(W) = \bigcap_{k \in N} F_k^{-1}(O_k) \in \Sigma$$

Conversely, let us take $t \in \bigcap_{k \in N} F_k^{-1}(O_k)$. Select a point $x_k \in F_k(t) \cap O_k$ for all $k \in N$.[1] Since $F(t)$ is weakly compact, the sequence $<x_k)$ has a weak cluster point $x \in G(t)$.

Moreover $\quad x \in \bigcap_{k \in N} \overline{O}_k = \bigcap_{k \in N} O_k = w.$

It follow that $t \in G^{-1}(w)$. This proves the claim and show that G is efforts measurable.

The theorem is proved.

Theorem 4.2 : Let $F :: \Omega \rightarrow k\,(X)$ be a scalary measurable multifunction. Then F admits a scalary measurable selector.

Proof : In order to prove this theorem, we follow below lemma and Lebesgue's dominated convergence theorem.

Lemma 4.2.1:

Given $C \in \cap\, wk\,(X)$ and $x^* \in X^*$.

We write $C/^{x^*} = \{x \in C : x^*(x) = \max x^*(C)\}$

$C/_{x^*} = \{x \in X : x^*(x) = \min x^*(C)\}$

Observe that $C/^{-x^*} = C/_{x^*}$ and that both $C/^{x^*}$ and $C/_{x^*}$ belong to $wk\,(x)$.

Let $F :: \Omega \rightarrow wk\,(X)$ be a scalary measurable multifunction and $x^* \in X^*$ then $F/^{x^*}$ and $F/_{x^*}$ are scalary measurable.

We divide the proof into two cases :

Particular case- Let us assume there is $M > 0$ such that for each $x^* \in Sx^*$, we have

$$|\,\delta *\,(x^*, F)\,| \leq M \quad \mu \; - a.e.$$

Clearly assumption ensures that for each $x^* \in Sx^*$, we have $|\,\delta *\,(x^*, F)\,| \leq M\ \mu\ - a.e.$ and that $\Delta F \leq 2M < \propto$. Let us define a sequence of scalary measurable multifunction $Fn :: \Omega \rightarrow k\,(X)$ with $F_n\,(t) \supset F_{n+1}\,(t)$ for every $n \in N$ and every $t \in \Omega$. Set $F_1 = F$ and if F_n is already defined then set $F_{n+1} = F_n/^{xn^*}$ where $x_n^* \in S_{x^*}$. is selected in such a way that

$$\int_w d^*\,(x_n^*, F_n) - d_*(x_n^*, F_n)\; dm^3\; DF_n/2 \qquad \text{... (4.2.1.1)}$$

By above lemma 4.2.1, each F_n is scalary measurable. The multifunction $G : \Omega \rightarrow k\,(X)$ given by $G\,(t) = \cap_{n \in N} F_n\,(t)$ is scalary measurable by following lemma 4.2.2.

Lemma 4.2.2 : $F_n : \Omega \rightarrow wk\,(X)$ be a sequence of scalary measurable multifunction such that $F_n\,(t) \supset F_{n+1}\,(t)$ for every $n \in N$ and every $t \in \Omega$. Then the multifunction $G : \Omega \rightarrow wk\,(X)$ given by $G\,(t) = \cap_{n \in N} F_n\,(t)$ is scalary measurable. And we have $G\,(t) \subset F\,(t)$ for all $t \in \Omega$.

In order to prove in the particular case we are dealing with it is sufficient to show that $\Delta G = 0$. We will prove this by contradiction.

Let us assume if it is possible that $\Delta G > 0$.

Then for each $n \in N$. We have $\Delta F_n \geq \Delta G > 0$ and equation (4.2.1.1) yields

$$\int_\Omega d^*(x^*, F_n) - d*(x_n^*, F_n)\; dm^3\; DG/2 > 0$$

By Lebesgue dominated convergence theorem, there is a point $t_0 \in \Omega$ at which the function $\delta^*(x^*, F_n) - \delta_*(x_n^*, F_n)$ does not tend to 0 as $n \rightarrow \propto$

Set $\varepsilon_n = \delta^*(x_n^*, F_n)\,(t_0) - \delta_*(x_n^*, F_n)\,(t_0)$ for every $n \in N$. By passing to a subsequence, we may assume that $\inf_{n \in N} \varepsilon_n = \varepsilon > 0$ for each $n \in N$.

We pick $x_n \in Fn\,(t_0)$ with $x_n{}^*(x_n) = \delta_*(x_n{}^*, F_n)\,(t_0)$.

Then given $m > n$, we have $x_m \in F_m\,(t_0) \subset F_{n+1}\,(t_0) = Fn \mid x_n{}^*(t_o)$

and so $$x_n{}^*(x_m) = \delta^*(x_n{}^*, F_n)\,(t_0).$$

hence $$\| x_m - x_n \| \geq x_n{}^*\,(x_m - x_n) = \varepsilon_n \geq \varepsilon.$$

Since all x_n's belong to the norm compact set $F\,(t_0)$.We reach a contradiction that finishes the proof of this case. **Proved**

Theorem 4.3 : Every scalary measurable multifunction $F : \Omega \to wk\,(X)$ admits a scalary measurable selector.

Proof : We will proof this theorem in two steps:

First step. We will prove this theorem by cotradiction. Let us suppose if it is possible there is $\varepsilon > 0$ such that $\Delta G > 0$ of every scalary measurable multifunction $G : \Omega \to wk\,(X)$ such that $G\,(t) \subset F\,(t)$ for all $t \in \Omega$.We define recurrently for each $\sigma \in \{0, 1\}^{< N}$ a functional $x_\sigma{}^* \in S_x{}^*$ and a scalary measurable multifunction $F_\sigma : \Omega \to wk\,(X)$ with $F_\sigma\,(t) \subset F\,(t)$ for all $t \in \Omega$.

Since $\Delta F > \varepsilon$, We can find $x^* \in S_x{}^*$ such that

$$\int_\Omega d^*(x^*, F) - \delta_*(x^*, F)\, d\mu > \varepsilon$$

Set $F\,(0) = F|_{x^*}$ and $F\,(1) = F|^{x^*}$ so that both $F(0)$ and $F(1)$ are scalary measurable by lemma 4.2.1

Assume that for some $\sigma \in \{0, 1\}^{<N}$ the multifunction F_σ is already constructed. Then $\Delta F_\sigma > \varepsilon$ and we can select $x_\sigma{}^* \in S_x{}^*$ such that

$$\int_\Omega \delta^*(x_\sigma{}^*, F_\sigma) - \sigma_*\,(x_\sigma{}^*, F_\sigma)\, d\mu > \varepsilon \qquad \text{... (4.3.1)}$$

Then we set $F_{\sigma\wedge 0} = F_\sigma|_{x^*\sigma}$ and $F_{\sigma\wedge 1} = F_\sigma|^{x^*\sigma}$ which are scalary measurable by lemma 4.2.1. Let us fix $n \in N$ and define the measurable function $a_n : \Omega \to R$ by

$$a_n\,(t) = 1/2^n \sum_{\sigma \in \{0,1\}^n} (\sigma^*(x_\sigma{}^*, F_\sigma)\,(t) - \sigma_*\,(x_\sigma{}^*, F_\sigma)\,(t))$$

Clearly for each $n \in N$ we have $| a_n | \leq 2M$ μ–*a.e*

Moreover given any $t \in \Omega$

$$\lim_{n\to\infty} a_n\,(t) = 0$$

By Lebesgue dominated convergence theorem we have

$$\lim_{n\to\infty} \int_\Omega a_n d\mu = 0$$

However for each $n \in N$, by inequality (4.2.1.1) implies that

$$\int_{\Omega} a_n \, d\mu = 1/2^n \sum_{\sigma \in \{0,1\}^n} \int_{\Omega} \delta^*(x_\sigma^*, F_\sigma) \, d\mu > \varepsilon$$

This contradiction finishes the proof of the first step.

Step 2: By the first step, we can find a scalary measurable multifunction $F_1 : \Omega \to wk(X)$ such that $F_1(t) \subset F(t)$ for all $t \in \Omega$ and $\Delta F_1 < 0$. Again the first step applied to F_1 ensures the existence of a scalary measurable multifunction $F_2 : \Omega \to wk(X)$ such that $F_2(t) \subset F_1(t)$ such that $t \in \Omega$ and $\Delta F_2 \leq 1/2$.

In this way we can find a sequence of scalary measurable multifunction $F_n : \Omega \to wk(X)$ with $\Delta F_n \leq 1/n$ such that $F_{n+1}(t) \subset F_n(t)$ for every $t \in \Omega$.

Then the multifunction $G : \Omega \to wk(X)$ given by $G(t) = \cap_{n \in N} F_n(t)$ is scalary measurable (by lemma 4.2.2) and $\Delta G = 0$ because $0 \leq \Delta G \leq \Delta Fn$ for every $n \in N$ consequently every selector of G (which in turn is a selector of F) is scalary measurable.

Theorem is proved.

Application: Application of this result is in the following area of mathematics:

(1) Control theory:

(2) Game theory :

(3) Differential inclusion:

(4) Mathematical models in economy.

(5) Integration of multifunction.

REFERENCES

1. Barabti, A. and Hess, C., The largest class of closed convex valved multifunction for which efforts measurability and scalar measurability coincide, *Set Valved Anal.*, **6,** No. **3**, 209-236 (1998).
2. Cascales, B., Kadets, V. and Rodriguez, J., Measurable selectors and Set valued pettis integral in non-separable Banach spaces, *J. Function Anal.*, **256**, No. **3**, 673-699 (2009).
3. Cascales, B., Kadets, V. and Rodriguez, J., Measurability and Selections of multifunction in Banach Space.
4. Chistyakov, V.V., Selections of bounded variations, *Journals of Applied Analysis*, Vol, **(10)**, No. **1,** pp 1-82 (2004).
5. Dragan Djurcie, Ljubisa, Kocinac, D. R., *A note on Convergence in Measure and Selections Principles*, Filon at 26/03/2012 pp 473-477 (2012).
6. Darinka, Dentcheva, *Regular Selections of Multi Functions and Random Sets* (2005).
7. Jayne, J.E, Orihuela, J., and Pallares, A.J., Vera, A., σ fragmentability of multivalved maps and selection theorems, *J. Funct. Anal*., **117,** No. **2**, pp 243-273 (1993).
8. Klein, E. and Thompson, A.C., *Theory of Correspondences Canadian Mathematical Society Series of Monographs and Advanced Texts*, Jhon Wiley & Sons Inc. Newyork (1984).

9. Kuratowski, K. and Ryll-Nardzewski, A genera theorem on Selectors, *Bull, Acad. Polon. Sci. Ser. Sci. Math. Astronom Phys*., **13** pp. 397-403 (1965).
10. Pandey, S.K. and Singh, D.K., Existence of Mesurable selectors in pettis integrable multi-functions. *J. Comp. and Maths. Sci*., Vol. **5 (1)**, pp 79-83 (2014).

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