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ON PARSEVAL'S EQUATION AND CONVOLUTION FOR FRACTIONAL HANKEL TRANSFORM

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Fractional Hankel transform which is generalization of Hankel transform is most applicable transform in optics, quantum mechanics etc. Its various properties are studied by number of mathematicians but its convolution is not yet discussed. In present paper, Parseval's equation in fractional Hankel transform is presented. Also convolution is defined for the fractional Hankel transform domain and corresponding convolution theorem is proved. Some properties of this convolution are discussed.

KEY WORDS: Fractional Hankel transform, Convolution, Parseval relation.

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INTRODUCTION

Hankel transform is widely applicable mathematical tool in Physics and applied sciences *e.g.* zero order Hankel transform describes diffraction effect of axial symmetric light beam in free space and higher order Hankel transform are used in the analysis of Laser cavity with circular motion. Fractional Hankel transform is generalization of Hankel transform in a fractional domain. It is well known that in case of rotational symmetry the fractional Fourier transform becomes Fractional Hankel transform. This connection opened the scope of applications of fractional Hankel transform in optics. Fan Ge *et al* [2] applied fractional Hankel transform in misaligned optical system.

Since Namias [4] developed fractional Fourier transform in 1980 and opened the new research area for the fractional integral transform, numbers of integral transforms are generalized in the fractional domain. Namias himself had introduced fractional Hankel transform in [5], but this transform is not the generalization of generalized Hankel transform, $H[f(x)](y) = \int_0^\infty \sqrt{xy}\, J_v(xy) f(x)dx, y \in (0,\infty)$, defined by Zemanian [12]. Further Kerr [3] had explored this in the fractional domain and defined fractional Hankel transform as,

$$H_\nu^\alpha\left[f(x)\right]\left(y\right)=\int_0^\infty f(x).K_\alpha(x,y)dx\,,$$

where
$$K_\alpha(x,y)=A_{\nu,\alpha}\exp\left(-\frac{i}{2}\left(x^2+y^2\right)\cot\frac{\alpha}{2}\right)\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)^{1/2}J_\nu\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)$$

$$=\delta\left(x-y\right)\ \text{for}\ \alpha=0\ \&\ \alpha=2\pi\,,$$

and
$$A_{\nu,\alpha}=\left|\sin\frac{\alpha}{2}\right|^{-1/2}\exp\left(i\left(\frac{\pi}{2}\hat{\alpha}-\frac{\alpha}{2}\right)\left(\nu+1\right)\right),\ \hat{\alpha}=\operatorname{sgn}\alpha\,,$$

$f\in L^2(R^+)$, $\alpha\in R$ and $\nu>-1$.

Prasad [6, 7, 8] studied fractional Hankel transform of tempered distribution. He had also introduced Pseudo differential operator involving fractional Hankel transform and investigated some properties. Zhang *et al* [11] generated the self fractional Hankel transform and studied its properties. We have also studied the analytic aspects of this transform and operation transform formulae in [9].

In the theory of integral transforms convolution has a special place. It helps in case of applications. But convolution of Hankel transform is not as simple as Laplace or Fourier because simple expression is not possible for the product $J_v(x)\,J_v(y)$ as $e^{-iux}e^{-ivx}$ in Fourier Transform. Tuan [10] in 1995 had discussed Hankel type convolution. Here we define convolution for fractional Hankel transform.

The paper is organized as follows. First of all in section II we have proved the Parseval's equation for fractional Hankel transform. Then convolution is defined and some of its properties are discussed in section III. Lastly convolution theorem is proved and the paper is concluded.

Parseval Relation

If $H_v^\alpha[f(x)]=F(y)$ and $H_v^\alpha[g(x)]=G(y)$, then

(a) $\int_0^\infty F(y)\overline{G(y)}dy=\int_0^\infty f(x)\overline{g(x)}\text{dx}$,

(b) $\int_0^\infty |F(y)|^2dy=\int_0^\infty |f(x)|^2\,dx$,

where bar over a function implies complex conjugate.

Proof:

(i) Let

$$H_v^\propto[f(x)](y)=F(y)=\int_0^\infty f(x)K_\propto(x,y)dx, \qquad \ldots(1)$$

$$H_v^\propto[g(x)](y)=G(y)=\int_0^\infty g(x)K_\propto(x,y) \qquad \ldots(2)$$

where
$$k_\propto(x,y)=A_{v,\propto}\,e^{-\frac{i}{2}(x^2+y^2)cot\frac{\propto}{2}}J_v\left(\frac{xy}{\left|\sin\frac{\propto}{2}\right|}\right)\sqrt{\frac{xy}{\left|\sin\frac{\propto}{2}\right|}}\,,$$

and $$A_{v,\ \propto} = \left|\sin\frac{\propto}{2}\right|^{-1/2} e^{i\left(\frac{\pi}{2}\hat{\propto}-\frac{\propto}{2}\right)},$$

and $$f(x) = \int_0^\infty \overline{K_\propto(x,y)}\ F(y)dy \qquad \dots (3)$$

$$g(x) = \int_0^\infty \overline{K_\propto(x,y)}\ G(y)dy. \qquad \dots (4)$$

Now (2) $$\Rightarrow \overline{G(y)} = \int_0^\infty \overline{g(x)}\ \overline{(K_\propto(x,y))}dx,$$

and (4) $$\Rightarrow \overline{g(x)} = \int_0^\infty \overline{G(y)}\ \overline{\overline{(K_\propto(x,y))}}dy. \qquad \dots (5)$$

Now $$\int_0^\infty F(y)\overline{G(y)}\,dy = \int_0^\infty\left[\int_0^\infty f(x)K_\propto(x,y)dx\right]\overline{G(y)}\,dy.$$

Interchanging the order of integration on the right side, using Fubinious theorem,

$$\int_0^\infty F(y)\overline{G(y)}\,dy = \int_0^\infty f(x)\left[\int_0^\infty \overline{G(y)}K_\propto(x,y)dy\right]dx$$

$$= \int_0^\infty f(x)\left[\int_0^\infty \overline{G(y)}\ \overline{\overline{K_\propto(x,y)}}dy\right]dx$$

$$= \int_0^\infty f(x)\overline{g(x)}\,dx, \text{ using (2).}$$

Hence proved.

(ii) Let $f(x) = g(x)$, $F(y) = G(y)$ and $\overline{F(y)} = \overline{G(y)}$, then above equation become,

$$\int_0^\infty F(y)\overline{F(y)}\,dy = \int_0^\infty f(x)\overline{f(x)}\,dx,$$

i.e., $$\int_0^\infty |F(y)|^2 dy = \int_0^\infty |f(x)|^2 dx.$$

Hence proved.

CONVOLUTION IN FRACTIONAL HANKEL TRANSFORM DOMAIN

Definition 3.1: We define the fractional Hankel type convolution, for any two functions *f*(*x*) and *g*(*x*), $0 < x < \infty$ as,

$$(f*g)(x) = h(x)$$

$$= \left|\sin\frac{\propto}{2}\right|^{v} e^{i\left(\frac{\pi}{2}\hat{\alpha}-\frac{\alpha}{2}\right)(v+1)} x^{-v+\frac{1}{2}} e^{\frac{i}{2}x^2 cot\frac{\propto}{2}} \frac{2^{1-3v}}{\sqrt{\pi}\,\Gamma\left(v+\frac{1}{2}\right)} \int_{(y+z)>x}\int_{|y-z|<x}[x^2 - (y-z)^2]^{v-\frac{1}{2}}[(y+z)^2-x^2]^{v-\frac{1}{2}}(yz)^{-v+\frac{1}{2}} f(y)g(z)e^{-\frac{i}{2}(y^2+z^2)\cot\frac{\propto}{2}}dydz,$$

$$x \in (0,\infty).$$

We observe the following property:

Commutativity:

$$(f*g)(x) = (g*f)(x),$$

$$(f*g)(x) = \left|\sin\frac{\propto}{2}\right|^{v} e^{i\left(\frac{\pi}{2}\hat{\alpha}-\frac{\alpha}{2}\right)(v+1)} x^{-v+\frac{1}{2}} e^{\frac{i}{2}x^2\cot\frac{\propto}{2}} \frac{2^{1-3v}}{\sqrt{\pi}\,\Gamma\left(v+\frac{1}{2}\right)} \int_{(y+z)>x}\int_{|y-z|<x}[x^2 - (y-z)^2]^{v-\frac{1}{2}}\ [(y+z)^2-x^2]^{v-\frac{1}{2}}(yz)^{-v+\frac{1}{2}} f(y)g(z)e^{-\frac{i}{2}(y^2+z^2)\cot\frac{\propto}{2}}dydz$$

$$= (g*f)(x).$$

Hence proved.

CONVOLUTION THEOREM

If $h(x) = \left|\sin\frac{\propto}{2}\right|^{v} e^{i\left(\frac{\pi}{2}\hat{\alpha}-\frac{\alpha}{2}\right)(v+1)} x^{-v+\frac{1}{2}} e^{\frac{i}{2}x^2\cot\left(\frac{\propto}{2}\right)} \frac{2^{1-3v}}{\sqrt{\pi}\Gamma\left(v+\frac{1}{2}\right)}$

$$\int_{u+w>x}\int_{|u-w|<x} [x^2-(u-w)^2]^{v-\frac{1}{2}}[(u+w)^2 - x^2]^{v+\frac{1}{2}}(uw)^{-v+\frac{1}{2}} f(u)g(w)e^{\frac{-i}{2}(u^2+w^2)\cot\frac{\propto}{2}}du dw, \text{x} \in (0,\infty),$$

where the function $h(x)$ is called the fractional Hankel convolution of the function $f(x)$ with $g(x)$. Then

$H_v^{\propto}[h(x)](y) = e^{\frac{i}{2}y^2\cot\frac{\propto}{2}} y^{-v} B_{\propto,v} H_v^{\alpha}[f(u)](y) H_v^{\alpha}[g(w)](y)$, where $B_{\propto,v} = \left|\sin\frac{\propto}{2}\right|^{2v-3}$.

Proof:

Using the result (31) on p. no. 52 of [1], i.e., $\int_0^\infty t^{\frac{1}{2}-v} J_v(xt)J_v(ut)J_v(wt)t^{\frac{1}{2}}x^{\frac{1}{2}}dt$

$$= \frac{[x^2-(u-w)^2]^{v-\frac{1}{2}}[(u+w)^2-x^2]^{v-\frac{1}{2}}}{x^{v-\frac{1}{2}}2^{3v-1}\sqrt{\pi}\,(uw)^v\Gamma\left(v+\frac{1}{2}\right)}, |u-w| < x < u+w,$$

Since $J_v(x) = \begin{cases} \sqrt{\frac{2}{\pi x}}\left(x - \cos\frac{\pi v}{2} - \frac{\pi}{4}\right) + O\left(x^{-\frac{3}{2}}\right), x \to +\infty. \\ O(x^v), \quad x \to +0 \end{cases}$

We have,

$$h(x) = \int_0^\infty \int_0^\infty \sqrt{uw}\, f(u)g(w)\left[\int_0^\infty t^{\frac{1}{2}-v} J_v(xt)J_v(ut)J_v(wt)t^{\frac{1}{2}}x^{\frac{1}{2}}dt\right] \left|\sin\frac{\propto}{2}\right|^{v} e^{i\left(\frac{\pi}{2}\hat{\alpha}-\frac{\alpha}{2}\right)(v+1)}\, e^{-\frac{i}{2}(u^2+w^2)cot\frac{\propto}{2}} e^{\frac{i}{2}x^2cot\frac{\propto}{2}}dudw.$$

Above equation may be arranged as,

$$h(x) = B_{\propto,v}\int_0^\infty\left[\left|\sin\frac{\propto}{2}\right|^{-\frac{1}{2}} e^{-i\left(\frac{\pi}{2}\hat{\alpha}-\frac{\alpha}{2}\right)(v+1)}\, e^{\frac{i}{2}(x^2+t^2)\cot\frac{\propto}{2}}\left(\frac{xt}{\left|\sin\frac{\propto}{2}\right|}\right)^{\frac{1}{2}} J_v\left(\frac{xt}{\left|\sin\frac{\propto}{2}\right|}\right)\right] e^{\frac{i}{2}t^2\text{Cot}\frac{\propto}{2}}$$

$$\int_0^\infty\left[\left|\sin\frac{\propto}{2}\right|^{-\frac{1}{2}} e^{i\left(\frac{\pi}{2}\hat{\alpha}-\frac{\alpha}{2}\right)(v+1)}\, e^{-\frac{i}{2}(u^2+t^2)\cot\frac{\propto}{2}}\left(\frac{ut}{\left|\sin\frac{\propto}{2}\right|}\right)^{\frac{1}{2}} J_v\left(\frac{ut}{\left|\sin\frac{\propto}{2}\right|}\right) f(u)du\right]$$

$$\int_0^\infty\left[\left|\sin\frac{\propto}{2}\right|^{-\frac{1}{2}} e^{i\left(\frac{\pi}{2}\hat{\alpha}-\frac{\alpha}{2}\right)(v+1)}\, e^{-\frac{i}{2}(w^2+t^2)\cot\frac{\propto}{2}}\left(\frac{wt}{\left|\sin\frac{\propto}{2}\right|}\right)^{\frac{1}{2}} J_v\left(\frac{wt}{\left|\sin\frac{\propto}{2}\right|}\right) f(w)dw\right] dt,$$

where $B_{\propto,v} = \left|\sin\frac{\propto}{2}\right|^{2v-3}$.

$$h(x) = \int_0^\infty \overline{K_\propto(x,t)}\{H_v^\alpha[f(u)](t)H_v^\alpha[g(w)](t)\}e^{\frac{i}{2}t^2\cot\frac{\propto}{2}}t^{-v}B_{\propto,v}dt$$

$$= (H_v^{\propto})^{-1}\left[e^{-\frac{i}{2}t^2\cot\frac{\propto}{2}}t^{-v}B_{\propto,v}H_v^{\alpha}[f(u)](t)H_v^{\alpha}[g(w)](t)\right]$$

$$[H_v^{\propto}h(x)](t) = e^{-\frac{i}{2}t^2 cot\frac{\propto}{2}}t^{-v}B_{\propto,v}H_v^{\alpha}[f(u)](t)H_v^{\alpha}[g(w)](t),$$

i.e., $[H_v^{\propto}h(x)](y) = e^{-\frac{i}{2}y^2 cot\frac{\propto}{2}}y^{-v}B_{\propto,v}H_v^{\alpha}[f(u)](y)H_v^{\alpha}[g(w)](y),$

is the required convolution of $f(x)$ with $g(x)$.

Conclusion

We have introduced the Parseval's equation and convolution theorem, for fractional Hankel transform and discussed one of its properties.

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