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ON SQUARE PENTAGONAL NUMBERS

P. JAYAKUMAR, P. REKA AND S. PRABHA

Deptt. of Math, A.V.V.M Sri Pushpam College (Autonomous), Poondi, Thanjavur (Dist.) - 613 503 (T.N), India

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A number which is simultaneously a pentagonal number P_l and a square number S_m.

Such numbers exist when

$$\frac{1}{2}l\{3l-1\}=m^2$$

A pentagonal square triangular number is a number that is simultaneously a pentagonal number P_l, a square number S_m, and a triangular number T_n. This requires a solution to the system of Diophantine equations

$$\frac{1}{2}l\{3l-1\}=m^2=\frac{1}{2}n(n+1)$$

Solutions of this system can be searched for by checking pentagonal triangular numbers (for which there is a closed-form solution) up to some limit to see if any are also square. Other than the trivial case $P_1=S_1=T_1=1$, using this approach shows that none of the first 9690 pentagonal triangular numbers are square, thus showing that there is no other pentagonal square triangular number less than $10^{22.166}$ (E. W. Weisstein, Sept. 12, 2003).

It is almost certain, therefore, that no other solution exists, although no proof of this fact appears to have yet appeared in print. However, recent work by J. Sillcox (pers. comm., Nov. 8, 2003 and Feb. 17, 2006) may have finally settled the problem. This work used a paper by Anglin (1996) that proves simultaneous Pell equations $x^2-Ry^2=1, z^2-Sy^2=1$ have exactly 19900 solutions with $R<S\le 200$. For example, if R = 11 and S = 56, then (199, 60, 449) is a solution. Sillcox then shows that the pentagonal square triangular number problem is equivalent to solving $x^2-2y^2=1$, $z^2-6y^2=1$, putting it within the bounds of Anglin's proof. For R = 2 and S = 6, only the trivial solution exists.

In this paper we report on the general form of the rank of square pentagonal numbers .The recurrence relations satisfied by the solutions are also given.

KEYWORDS: The rank of square pentagonal numbers, the recurrence relations satisfied by the solutions.

Subject Classifications: MSC: 11A, 11D

Introduction

The theory of numbers has been called the Queen of Mathematics because of its rich varieties of fascinating problems. Many numbers exhibit fascinating properties, they form sequences, they form patterns and so on. A pentagonal number is a figurate number that extends the concept of triangular and square numbers to the pentagon, but, unlike the first two, the patterns involved in the construction of pentagonal numbers are not rotationally symmetrical. The n^{th} pentagonal number p_n is the number of distinct dots in a pattern of dots consisting of the outlines of regular pentagons whose sides contain 1 to n dots, overlaid so that they share one vertex. For instance, the third one is formed from outlines comprising 1, 5 and 10 dots, but the 1, and 3 of the 5, coincide with 3 of the 10 – leaving 12 distinct dots, 10 in the form of a pentagon, and 2 inside…

P_n is given by the formula $\quad P_n = \frac{1}{2}n(3n-1)$ for $n \geq 1$.

The first few pentagonal numbers are: 1, 5, 12, 22, 35,…. We determine the general form of the rank of square pentagonal numbers. Also the recurrence relations satisfied by the solutions are presented.

Theorem

The general form of the ranks of square Pentagonal number (P_m) are given by

$$m = \frac{1}{12}\left[(5+2\sqrt{6})^{2r+1} + (5-2\sqrt{6})^{2r+1} + 2\right], \text{ where } r = 0, 1, 2, ...$$

Proof : Let P_m be a square Pentagonal number. We write

$$P_m = t^2, \text{ where } t \text{ is a non-zero integer.} \qquad \ldots(1)$$

Using the definition of Pentagonal number the above equation (1) is written as

$$m\,(3m-1) = 2t^2.$$

$$3\left(m^2 - \frac{m}{3}\right) = 2t^2.$$

By writing complete square, we get

$$3\left[\left(m - \frac{1}{6}\right)^2 - \frac{1}{36}\right] = 2t^2.$$

$$3\left[\frac{(6m-1)^2}{36} - \frac{1}{36}\right] = 2t^2.$$

On simplifying we get,

$$(6m-1)^2 - 1 = 24t^2 \text{ (or) } (6m-1)^2 - 24t^2 = 1.$$

If we take $\quad x = 6m - 1 \qquad \ldots(2)$

then we get $x^2 - 24t^2 = 1,$ . ..(3)

which is the well-known Pell's equation whose solutions are given by

$$x_n = \frac{1}{2}\left[\left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1}\right], \quad ...(4)$$

$$t_n = \frac{1}{4\sqrt{6}}\left[\left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}\right], \quad ...(5)$$

where $n = 0, 1, 2, \ldots$

In view of the equation (2), the rank m of square Pentagonal number and the values of t are given by

$$m = \frac{1}{12}\left[\left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1} + 2\right], \quad ... (6)$$

$$t = \frac{1}{4\sqrt{6}}\left[\left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}\right]. \quad \ldots (7)$$

It is noted that the values of ranks m and t will be in integer only when n are taking even values. Thus the ranks m of the square Pentagonal number P_m and the values of t are given by

$$m = \frac{1}{12}\left[\left(5+2\sqrt{6}\right)^{2r+1} + \left(5-2\sqrt{6}\right)^{2r+1} + 2\right], \quad ...(8)$$

$$t = \frac{1}{4\sqrt{6}}\left[\left(5+2\sqrt{6}\right)^{2r+1} - \left(5-2\sqrt{6}\right)^{2r+1}\right], \quad ...(9)$$

where $r = 0$, 1, 2,…

For simplicity and brevity some values of m, t and their corresponding pentagonal and square number s are presented in the following table.

Table

Values of *r*	**Ranks (*m*)**	**Ranks (*t*)**	**Pentagonal numbers** (P_m)	**Square numbers** (t^2)
0	1	1	1	1
2	81	99	9801	9801
4	7921	9701	94109401	94109401
6	776161	950599	903638458801	903638458801
8	76055841	93149001	868428229472001	868428229472001

It is interesting to note that the ranks of the square pentagonal numbers are also squares.

Theorem: The ranks m_r and t_r of the equations (6) and (7) satisfy the following recurrence relations:

(i) $m_{2r+4} - 98m_{2r+2} + m_{2r} = -16,$

(ii) $t_{2r+4} - 98t_{2r+2} + t_{2r} = 0,$

where m_r and t_r are the ranks of the r th -square pentagonal number and squares number.

Proof of (i)

Let $$m_{2r} = \frac{1}{12}\left[\left(5+2\sqrt{6}\right)^{2r+1} + \left(5-2\sqrt{6}\right)^{2r+1} + 2\right]. \quad \ldots(10)$$

This equation (10) can be written as

$$12m_{2r} - 2 = \left[\left(5+2\sqrt{6}\right)^{2r+1} + \left(5-2\sqrt{6}\right)^{2r+1}\right] \quad \ldots(11)$$

If we take $A = 5+2\sqrt{6}$, then $A-4\sqrt{6} = 5-2\sqrt{6}$ and the equation (11) becomes

$$12m_{2r} - 2 = A^{2r+1} + (A-4\sqrt{6})^{2r+1} \quad \ldots(12)$$

Replacing r by $r+1$, $r+2$ successively in (12), we get

$$12m_{2r+2} - 2 = A^{2r+3} + (A-4\sqrt{6})^{2r+3}, \quad \ldots(13)$$

$$12m_{2r+5} - 2 = A^{2r+5} + (A-4\sqrt{6})^{2r+5}. \quad \ldots(14)$$

Multiplying the equation (12) by A^2, and then subtracting from the equation (13) we get

$$[12m_{2r+2} - 2] - [12m_{2r} - 2]A^2 = (A-4\sqrt{6})^{2r+1}[(A-4\sqrt{6})^2 - A^2] \quad \ldots(15)$$

Multiplying the equation (13) by A^2, and then subtracting from the equation (14) we get,

$$[12m_{2r+4} - 2] - [12m_{2r+2} - 2]A^2 = (A-4\sqrt{6})^{2r+3}[(A-4\sqrt{6})^2 - A^2] \quad \ldots(16)$$

Multiplying the equation (15) by $(A-4\sqrt{6})^2$, and then subtracting from the equation (16) we get,

$$[12m_{2r+4} - 2] - [12m_{2r+2} - 2]A^2 - [12m_{2r+2} - 2](A-4\sqrt{6})^2 + [12m_{2r} - 2]A^2(A-4\sqrt{6})^2 = 0$$

$$[12m_{2r+4} - 2] - [12m_{2r+2} - 2][A^2 - (A-4\sqrt{6})^2] + [12m_{2r} - 2]A^2(A-4\sqrt{6})^2 = 0$$

$$[12m_{2r+4} - 2] - [12m_{2r+2} - 2][(5+2\sqrt{6})^2 - (5-2\sqrt{6})^2] + [12m_{2r} - 2](5+2\sqrt{6})^2(5-2\sqrt{6})^2 = 0$$

On simplifying, we get

$$[12m_{2r+4} - 2] - 98\,[12m_{2r+2} - 2] + [12m_{2r} - 2] = 0$$

$$12m_{2r+4} - 1176m_{2r+2} + 12m_{2r} = -192$$

On dividing by 12, we get

$$m_{2r+4} - 98\,m_{2r+2} + m_{2r} = -16. \quad \ldots(17)$$

This is the recurrence relation satisfied by the ranks *m*.

It is observed that the values of *m* : (1, 81, 7921); (81, 7921, 776161); and (7921, 776161, 76055841) are satisfied by the equation (17).

Proof of (ii)

Let $$t_{2r} = \frac{1}{4\sqrt{6}}\left[\left(5+2\sqrt{6}\right)^{2r+1} - \left(5-2\sqrt{6}\right)^{2r+1}\right] \quad \ldots(18)$$

This equation (18) can be written as

$$4\sqrt{6}\,t_{2r} = \left[\left(5+2\sqrt{6}\right)^{2r+1} - \left(5-2\sqrt{6}\right)^{2r+1}\right] \quad \ldots(19)$$

If we take $A = 5+2\sqrt{6}$, then $A-4\sqrt{6} = 5-2\sqrt{6}$ and the equation (19) becomes

$$4\sqrt{6}\,t_{2r} = A^{2r+1} - (A-4\sqrt{6})^{2r+1} \quad \ldots(20)$$

Replacing r by $r+1$, $r+2$ successively in (20), we get

$$4\sqrt{6}\,t_{2r+2} = A^{2r+3} - (A-4\sqrt{6})^{2r+3}, \quad \ldots(21)$$

$$4\sqrt{6}\,t_{2r+4} = A^{2r+5} - (A-4\sqrt{6})^{2r+5}. \quad \ldots(22)$$

Multiplying the equation (20) by A^2, and then subtracting from the equation (21) we get

$$4\sqrt{6}\,t_{2r+2} - 4\sqrt{6}\,t_{2r}\,A^2 = (A-4\sqrt{6})^{2r+1}[A^2 - (A-4\sqrt{6})^2] \quad \ldots(23)$$

Multiplying the equation (21) by A^2, and then subtracting from the equation (22) we get,

$$4\sqrt{6}\,t_{2r+4} - 4\sqrt{6}\,t_{2r+2}\,A^2 = (A-4\sqrt{6})^{2r+3}[A^2 - (A-4\sqrt{6})^2] \quad \ldots(24)$$

Multiplying the equation (23) by $(A-4\sqrt{6})^2$, and then subtracting from the equation (24) we get

$$4\sqrt{6}\,t_{2r+4} - 4\sqrt{6}\,t_{2r+2}\,[A^2 - (A-4\sqrt{6})^2] + 4\sqrt{6}\,t_{2r}A^2(A-4\sqrt{6})^2 = 0 \quad \ldots(25)$$

On dividing by $4\sqrt{6}$, we get

$$t_{2r+4} - t_{r+2}\,[A^2 - (A-4\sqrt{6})^2] + t_{2r}A^2(A-4\sqrt{6})^2 = 0, \text{ where } A = 5+2\sqrt{6}$$

$$t_{2r+4} - t_{2r+2}\,[(5+2\sqrt{6})^2 - (5-2\sqrt{6})^2] + t_{2r}A^2(A-4\sqrt{6})^2 = 0.$$

$$t_{2r+4} - 98t_{2r+2} - t_{2r} = 0 \quad \ldots(26)$$

This is the recurrence relation satisfied by the ranks *t*.

It is observed that the values of *t* : (1, 99, 9701); (99, 9701, 950599); and (9701, 950599, 93149001) are satisfied by the equation (26).

Conclusion

One may search for other integral solutions of (8), and (9).

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