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SOME CLASSES OF ENTIRE SEQUENCE OF INTERVAL NUMBERS

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In this paper we introduce the new concept of interval valued sequence space $G^{i}_{\lambda^2}$ where (λ_k) is a fixed sequence of positive real numbers. We study different topological properties of this space completeness, solidity, AB property, sequence algebra etc. Also we obtain some inclusion relation involving this sequence space.

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INTRODUCTION

Interval arithmetic was first suggested by Dwyer [6] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore [12] in 1959 and Moore and Yang [13] 1962. Furthermore, Moore and others [7] and [14] have developed applications to differential equations.

Chiao in [10] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Sengönül and Eryilmax [15] in 2010 introduced and studied bounded and convergent sequence space of interval numbers and showed that these spaces are complete metric space. Recently Esi [1], [2], [3], [4] and [9] introduced some new type sequence spaces of interval numbers.

A set consisting of a closed interval of real numbers x such that $a \le x \le b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties. We denote the set of all real valued closed intervals by $I\Re$. Any elements of $I\Re$ is called closed interval and denoted by $\bar{x}$. That is $\bar{x} = \{x \in \Re : a \le x \le b\}$. An interval number $\bar{x}$ is a closed subset of real numbers. Let x_l and x_r be be respectively first and last points of the interval number $\bar{x}$.

For $\bar{x}_1, \bar{x}_2 \in I\Re$, we define $\bar{x}_1 = \bar{x}_2$ if and only if $x_{1l} = x_{2l}$ and $x_{1r} = x_{2r}$

$$\bar{x}_1 + \bar{x}_2 = \{x \in \Re : x_{1l} + x_{2l} \le x \le x_{1r} + x_{2r})\}$$

$$\bar{x}_1 \times \bar{x}_2 = \{x \in \Re : \min(x_{1l}x_{2l},\, x_{1l}x_{2r},\, x_{1r}x_{2l},\, x_{1r}x_{2r}) \le x \le \max(x_{1l}x_{2l},\, x_{1l}x_{2r},\, x_{1r}x_{2l},\, x_{1r}x_{2r})\}$$

The set of all interval numbers $I\Re$ is a complete metric space defined by

$$d(\bar{x}_1, \bar{x}_2) = \max\{|\bar{x}_{1l} - \bar{x}_{2l}|, |\bar{x}_{1r} - \bar{x}_{2r}|\}$$

In the special case $\bar{x}_1 = [a, a]$ and $\bar{x}_2 = [b, b]$, we obtain usual metric of $\Re$.

Let us define transformation $f : N \to \Re$, $k \to f(k) = \bar{x}_k$, then $\bar{x} = (\bar{x}_k)$ is called sequence of interval numbers. $\bar{x}_k$ is called k^{th} term of sequence $\bar{x} = (\bar{x}_k)$, ω^i denotes the set of all interval numbers with real terms and the algebraic properties of ω^i are in [11].

A sequence $\bar{x} = (\bar{x}_k)$ of interval numbers is said to be convergent to the interval number $\bar{x}_0$ if for each $\varepsilon > 0$ there exists a positive integer k_0 such that $d(\bar{x}_k, \bar{x}_0) < \varepsilon$ for all $k \ge k_0$ and we denote it by $\lim_k \bar{x}_k = \bar{x}_0$. Equivalently $\lim_k \bar{x}_k = \bar{x}_0$ iff $\lim_k x_{kl} = x_{0l}$ and $\lim_k x_{kr} = x_{0r}$.

An interval valued sequence space E^i is said to be solid if $\bar{y} = (\bar{y}_k) \in E^i$ whenever $|\bar{y}_k| \le |\bar{x}_k|$ for all $k \in N$ and $\bar{x} = (\bar{x}_k) \in E^i$.

An interval valued sequence space E^i is said to be monotone if E^i contains the canonical pre-image of all its step spaces.

A interval sequence space E^i is said to be sequence algebra if $\bar{x} \otimes \bar{y} = (\bar{x}_k \otimes \bar{y}_k) \in E^i$, whenever $\bar{x} = (\bar{x}_k) \in E^i$, $\bar{y} = (\bar{y}_k) \in E^i$ [1].

Let us denote the space of all entire functions of interval numbers by Γ^i.

For each fixed k, we define the metric

$$\rho(\bar{x}_k, \bar{y}_k) = \max\{|x_{kl} - y_{kl}|^{1/k},\ |x_{kr} - y_{kr}|^{1/k}\} = [d(\bar{x}_k, \bar{y}_k)]^{1/k}$$

We define Γ^i by $\quad \Gamma^i = \{\bar{x} = (\bar{x}_k) \in \omega^i : \lim_{k\to\infty} \rho(\bar{x}_k, \bar{0}) = \bar{0}\}$

Throughout this paper, let $\lambda = (\lambda_k)$ be a fixed sequence of positive real numbers such that $\frac{\lambda_{k+1}}{\lambda_k} \to 1$ as $k \to \infty$ and $\lambda_k \ne 1$ for all k. The space $G^i_{\lambda^2}$ is defined by

$$G^i_{\lambda^2} = \{\bar{x} = (\bar{x}_k) : \sum_{k=1}^{\infty} \lambda_k^2 d(\bar{x}_k, \bar{0})^2 < \infty\}$$

Example: Let $\lambda = (\lambda_k) = (k), k \in N$ and $\bar{x} = (\bar{x}_k) = \left(\left[\frac{1}{k^4}, \frac{1}{k^2}\right]\right)$

Then
$$\sum_{k=1}^{\infty} \lambda_k^2 d(\bar{x}_k, \bar{0})^2 = \sum_{k=1}^{\infty} \lambda_k^2 \left[\max\left(\left|\frac{1}{k^4}\right|, \left|\frac{1}{k^2}\right|\right)\right]^2$$

$$= \sum_{k=1}^{\infty} k^2 \frac{1}{k^4} = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

Hence $(\bar{x}_k)$ is in $G^i_{\lambda^2}$.

The object of this paper is to investigate some properties of $G^i_{\lambda^2}$.

MAIN RESULTS

Theorem 2.1. The sequence space $G^i_{\lambda^2}$ is a complete metric space with respect to the metric defined by

$$\bar{d}(\bar{x}, \bar{y}) = \sum_{k=1}^{\infty} \lambda_k^2 d(\bar{x}_k, \bar{y}_k)^2 \quad \ldots (2.1)$$

Proof: Let $(\bar{x}^n)$ be a Cauchy sequence in $G^i_{\lambda^2}$. Then for a given $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$\bar{d}(\bar{x}^n, \bar{x}^m) < \varepsilon \quad \text{for all } n, m \geq n_0$$

then

$$\sum_{k=1}^{\infty} \lambda_k^2 d(\bar{x}_k^n, \bar{x}_k^m)^2 < \varepsilon \text{ for all } n, m \geq n_0 \quad \ldots (2.2)$$

$$d(\bar{x}_k^n, \bar{x}_k^m)^2 \lambda_k^2 < \varepsilon \text{ for all } n, m \geq n_0$$

$$d(\bar{x}_k^n, \bar{x}_k^m)^2 < \frac{\varepsilon}{\lambda_k^2} \text{ for all } n, m \geq n_0 \text{ and for all } k \in N$$

$$d(\bar{x}_k^n, \bar{x}_k^m) < \left(\frac{\varepsilon}{\lambda_k^2}\right)^{1/2} < \varepsilon \text{ for all } n, m \geq n_0 \text{ and for all } k \in N$$

This means that $(\bar{x}_k^n)$ is a Cauchy sequence in $I\Re$. Since $I\Re$ is a Banach space, $(\bar{x}_k^n)$ is convergent. Now, let $\lim_n \bar{x}_k^n = \bar{x}_k$ for each $k \in N$ and $\bar{x} = (\bar{x}_k)$.

Taking limit as $m \to \infty$ in (2.2) we have

$$\sum_{k=1}^{\infty} \lambda_k^2 d(\bar{x}_k^n, \bar{x})^2 < \varepsilon \text{ for all } n \geq n_0$$

$$\bar{d}(\bar{x}^n, \bar{x}) < \varepsilon \text{ for all } n \geq n_0$$

Now for all $n \geq n_{0,}$

$$\bar{d}(\bar{x}, 0) \leq \bar{d}(\bar{x}^n, \bar{x}) + \bar{d}(\bar{x}^n, 0) < \varepsilon + \infty = \infty$$

Thus $\bar{x} = (\bar{x}_k) \in G^i_{\lambda^2}$ and so $G^i_{\lambda^2}$ is complete. This completes the proof.

Theorem 2.2. $G^i_{\lambda^2}$ is a subset of Γ^i

Proof: Let $\bar{x} \in G^i_{\lambda^2}$, then $\sum_{k=1}^{\infty} \lambda_k{}^2 d(\bar{x}_k, \bar{0})^2 < \infty$ … (2.1)

where $$\frac{\lambda_{k+1}}{\lambda_k} \to 1 \text{ as } k \to \infty \text{ and } \lambda_k \neq 1 \text{ for all } k \quad \text{… (2.2)}$$

We claim that $[d(\bar{x}_k, \bar{0})]^{1/k}$ converges to zero as $k \to \infty$.

From Equation (2.1)

$$\lambda_k{}^2 d(\bar{x}_k, \bar{0})^2 < \varepsilon^{2k} \text{ for all } k \in N$$

$$\Rightarrow d(\bar{x}_k, \bar{0})^2 < \varepsilon^{2k} / \lambda_k{}^2$$

$$\Rightarrow d(\bar{x}_k, \bar{0}) < \varepsilon^k / \lambda_k$$

$$\Rightarrow [d(\bar{x}_k, \bar{0})]^{1/k} < \varepsilon / \lambda_k{}^{1/k} < \varepsilon_1 \quad \text{from (2.2)}$$

Hence $[d(\bar{x}_k, \bar{0})]^{1/k} \to 0$ as $k \to \infty$ and so $\bar{x} \in \Gamma^i$

Consequently, $G^i_{\lambda^2}$ is a subset of Γ^i.

Remark. $G^i_{\lambda^2}$ is a Banach space with norm

$$\|\bar{x}\|_{G^i_{\lambda^2}} = \left\{ \sum_{k=1}^{\infty} \lambda_k{}^2 [d(\bar{x}_k, \bar{0})]^2 \right\}^{1/2}$$

Theorem 2.3. If $G^i_{\lambda^2}$ and $G^i_{\mu^2}$ are two sequences of interval numbers, then $G^i_{\lambda^2} = G^i_{\mu^2}$ if and only if $k_1 \leq \frac{\lambda_k}{\mu_k} \leq k_2$, where k_1 and k_2 are constants.

Proof: The sufficiency of the condition $k_1 \leq \frac{\lambda_k}{\mu_k} \leq k_2$ … (2.3)

If $\lambda_k \leq k_2 \mu_k$ then $\lambda_k{}^2 d(\bar{x}_k, \bar{0})]^2 \leq k_2{}^2 \mu_k{}^2 d(\bar{x}_k, \bar{0})]^2$

If $(\bar{x}_k) \in G^i_{\mu}$, $\sum_{k=1}^{\infty} \mu_k{}^2 d(\bar{x}_k, \bar{0})^2 < \infty$

Therefore $$\sum_{k=1}^{\infty} \lambda_k{}^2 d(\bar{x}_k, \bar{0})^2 \leq \sum_{k=1}^{\infty} k_2{}^2 \mu_k{}^2 d(\bar{x}_k, \bar{0})^2 < \infty$$

This implies that $(\bar{x}_k) \in G^i_{\lambda^2}$

Hence $G^i_{\mu^2} \subset G^i_{\lambda^2}$ … (2.4)

Similarly, if $k_1 \mu_k \leq \lambda_k$ then $G^i_{\lambda^2} \subset G^i_{\mu^2}$ … (2.5)

From (2.4) and (2.5), $G^{i}_{\lambda^2} = G^{i}_{\mu^2}$

To prove the necessity of the condition, let us suppose that the condition is not satisfied. First consider the right hand side inequality of (2.3). Let $\frac{\lambda_k}{\mu_k} \to \infty$ as $k \to \infty$.

Then it has a subsequence $\frac{\lambda_{k_n}}{\mu_{k_n}} \to \infty$ as $k_n \to \infty$ in such a manner that $\frac{\lambda_{k_n}}{\mu_{k_n}} > n$ for the values $n = 1, 2, ...$ and $k_1 < k_2 < .....$

Now we shall define a sequence $(\bar{x}_k)$ as follows

$$\bar{x}_k = \begin{cases} \left[0, \frac{1}{n\ \mu_k}\right] & \text{when } k = k_n \\ [0,0] & \text{when } k \neq k_n \end{cases}$$

Then $$\sum_{k=1}^{\infty} \mu_k{}^2 d(\bar{x}_k, \bar{0})^2 = \sum_{n=1}^{\infty} \mu_{k_n}{}^2 d(\bar{x}_{k_n}, \bar{0})^2$$

$$= \sum_{n=1}^{\infty} \frac{\mu_{k_n}{}^2}{n^2 \mu_{k_n}{}^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Therefore $$(\bar{x}_k) \in G^{i}_{\mu^2} \qquad \dots (2.6)$$

But $$\sum_{k=1}^{\infty} \lambda_k{}^2 d(\bar{x}_k, \bar{0})^2 = \sum_{n=1}^{\infty} \lambda_{k_n}{}^2 d(\bar{x}_{k_n}, \bar{0})^2$$

$$> \sum_{n=1}^{\infty} n^2 \mu_{k_n}{}^2 d(\bar{x}_{k_n}, \bar{0})^2 = \sum_{n=1}^{\infty} \frac{n^2 \mu_{k_n}}{n^2 \mu_{k_n}} = \infty$$

Thus $$\sum_{k=1}^{\infty} \lambda_k{}^2 d(\bar{x}_k, \bar{0})^2 > \infty$$

Therefore $$(\bar{x}_k) \notin G^{i}_{\lambda^2} \qquad \dots (2.7)$$

From (2.6) and (2.7) contradict (2.4).

Similarly , if the left hand side inequality of (2.3) is not satisfied, then we can contradict (2.5) by constructing a sequence of the above type.

Hence the condition $k_1 \leq \frac{\lambda_k}{\mu_k} \leq k_2$ is necessary and sufficient in order that $G^{i}_{\lambda^2} = G^{i}_{\mu^2}$.

Theorem 2.4. $G^{i}_{\lambda^2}$ is an AK space.

Proof: For each $(\bar{x}_k) \in G^{i}_{\lambda^2}$. $\left\|(\bar{x}^{[n]}) - \bar{x}\right\| \to 0$ as $n \to \infty$

Hence $G^i_{\lambda^2}$ has *AK*.

Theorem 2.5. $G^i_{\lambda^2}$ has *AB* property.

Proof: It is enough to show that $G^i_{\lambda^2}$ has monotone norm. Indeed for $n < m$ and for every $(\bar{x}_k) \in G^i_{\lambda^2}$, we have $\left\|(\bar{x}^{[n]})\right\|^2 = \sum_{k=1}^{n} \lambda_k{}^2 d(\bar{x}_k, \bar{0})^2 < \sum_{k=1}^{m} \lambda_k{}^2 d(\bar{x}_k, \bar{0})^2 = \left\|(\bar{x}^{[m]})\right\|^2$

$$\left\|(\bar{x}^{[n]})\right\| < \left\|(\bar{x}^{[m]})\right\|$$

Also $\{\left\|(\bar{x}^{[n]})\right\|,\ n = 1, 2, ...\}$ is a monotonically increasing sequence of interval numbers bounded above by $\|\bar{x}\|_{G^i_{\lambda^2}}$

Hence $$\|\bar{x}\|_{G^i_{\lambda^2}} = \lim_{n\to\infty} \left\|(\bar{x}^{[n]})\right\| = \sup_n \{\left\|(\bar{x}^{[n]})\right\|, n = 1, 2, ...\}$$

Thus $G^i_{\lambda^2}$ has monotone norm.

Theorem 2.6. The space $G^i_{\lambda^2}$ is solid.

Proof: Let $(\bar{x}_k)$ and $(\bar{y}_k)$ be two sequences such that

$$(\bar{x}_k) \in G^i_{\lambda^2} \text{ and } d(\bar{y}_k, \bar{0}) \le d(\bar{x}_k, \bar{0}) \text{ for all } k \in N$$

Since $(\bar{x}_k) \in G^i_{\lambda^2}$, we have $\sum_{k=1}^{\infty} \lambda_k{}^2 d(\bar{x}_k, \bar{0})^2 < \infty$

Also we have $$\lambda_k{}^2 d(\bar{y}_k, \bar{0})^2 \le \lambda_k{}^2 d(\bar{x}_k, \bar{0})^2$$

$$\sum_{k=1}^{\infty} \lambda_k{}^2 d(\bar{y}_k, \bar{0})^2 \le \sum_{k=1}^{\infty} \lambda_k{}^2 d(\bar{x}_k, \bar{0})^2 < \infty$$

So $(\bar{y}_k) \in G^i_{\lambda^2}$. Therefore $G^i_{\lambda^2}$ is solid.

Theorem 2.7. The space $G^i_{\lambda^2}$ is symmetric.

Proof: Let $(\bar{x}_k)$ be a sequence in $G^i_{\lambda^2}$. Then $\sum_{k=1}^{\infty} \lambda_k{}^2 d(\bar{x}_k, \bar{0})^2 < \infty$

For $\varepsilon > 0$ there exists $k = k_0(\varepsilon)$ such that $\sum_{k=1}^{\infty} \lambda_k{}^2 d(\bar{x}_k, \bar{0})^2 - \sum_{k \le k_0} \lambda_k{}^2 d(\bar{x}_k, \bar{0})^2 < \varepsilon$

Let $(\bar{y}_k)$ be a rearrangement of $(\bar{x}_k)$ and k_1 be such that

$$\{\bar{x}_k : k \leq k_0\} \subseteq \{\bar{y}_k : k \leq k_1\}$$

Then $$\sum_{k=1}^{\infty} \lambda_k{}^2 d(\bar{y}_k, \bar{0})^2 - \sum_{k \leq k_1}^{\infty} \lambda_k{}^2 d(\bar{x}_k, \bar{0})^2 < \varepsilon$$

and so $$\sum_{k=1}^{\infty} \lambda_k{}^2 d(\bar{y}_k, \bar{0})^2 < \infty$$

Hence $(\bar{y}_k) \in G^i_{\lambda^2}$ and $G^i_{\lambda^2}$ is symmetric.

Theorem 2.8. The space $G^i_{\lambda^2}$ is sequence algebra.

Proof: We consider the space $G^i_{\lambda^2}$

Let $(\bar{x}_k)$ and $(\bar{y}_k)$ be two sequences in $G^i_{\lambda^2}$ and $0 < \varepsilon < 1$.

Then the result follows from the following inclusion relation.

$$\left\{k \in N : \bar{d}(\bar{x}_k \otimes \bar{y}_k, \bar{0})\right\} \supseteq \left\{k \in N : \bar{d}(\bar{x}_k, \bar{0})\right\} \cap \left\{k \in N : \bar{d}(\bar{y}_k, \bar{0})\right\}$$

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