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ON A SPECIAL RECTANGULAR PROBLEM

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In this paper we report on an infinite sequence of rectangles where in each of the addition of the area and perimeter are expressed as the difference of two squares. The recurrence relations satisfied by the solutions are presented.

KEY WORDS: Infinite sequence of rectangles, the area, the perimeter and the recurrence relations.

SUBJECT CLASSIFICATION: MSC: 11A, 11D

Introduction

In Euclidean plane geometry, a **rectangle** is any quadrilateral with four right angles. It can also be defined as an equiangular quadrilateral, since equiangular means that all of its angles are equal (360°/4 = 90°). It can also be defined as a parallelogram containing a right angle. A rectangle with four sides of equal length is a square. The term **oblong** is occasionally used to refer to a non-square rectangle. A rectangle with vertices *ABCD* would be denoted as ▭ *ABCD*.

The word rectangle comes from the Latin *rectangulus*, which is a combination of *rectus* (right) and *angulus* (angle).

A so-called **crossed rectangle** is a crossed (self-intersecting) quadrilateral which consists of two opposite sides of a rectangle along with the two diagonals It is a special case of an antiparallelogram, and its angles are not right angles. Other geometries, such as spherical, elliptic, and hyperbolic, have so-called rectangles with opposite sides equal in length and equal angles that are not right angles.

Consider two numbers 2 and 7. Now $2(2+7)+(2\times 7)=6^2-2^2$. If 2 and 7 are taken as the two sides of a rectangle then the above equation implies that the sum of the area and perimeter may be written as the difference of two squares. In [3] we have searched an infinite sequence of rectangles of sides x and y such that

$$(x\,y)-2\,(x+y)=x^2-y^2,$$

In this communication, we search an infinite sequence of rectangles of sides x and y such that

$$(x\,y)+2\,(x+y)=x^2-y^2,$$

as a particular problem of the sum of the area and perimeter is expressed as the difference of the two squares of two sides x and y.

Method of analysis

The equation to be solved is

$$(x\,y)+2\,(x+y)=x^2-y^2. \qquad \text{... (1)}$$

Introducing the linear transformation

$$x=u+v,\ y=u-v,\ \text{where } u>v, \qquad \text{... (2)}$$

the equation (1) becomes

$$(u+v)(u-v)+2[(u+v)+(u-v)]=(u+v)^2-(u-v)^2$$

$$u^2-v^2+4u=u^2+v^2+2uv-u^2-v^2+2uv$$

$$4u+u^2=4uv+v^2$$

$$=(v+2u)^2-4u^2$$

$$4u+5u^2=(v+2u)^2$$

$$5\left(u^2+\frac{4}{5}u\right)=(v+2u)^2$$

By writing complete squares, we get

$$5\left[\left(u+\frac{2}{5}\right)^2-\frac{4}{25}\right]=(v+2u)^2$$

On simplifying, we get

$$(5u+2)^2-4=5(v+2u)^2$$

Putting $X=v+2u$ and $Y=5u+2$ in the above equation, we obtain … (3)

$$Y^2-4=5X^2$$

i.e.,

$$Y^2-5X^2=4=2^2, \qquad \text{… (4)}$$

which is the well-known Pell's equation whose solutions [2] are given by

$$Y_n=(9+4\sqrt{5})^{n+1}+(9-4\sqrt{5})^{n+1}, \qquad \text{… (5)}$$

$$X_n=\frac{1}{\sqrt{5}}[(9+4\sqrt{5})^{n+1}-(9-4\sqrt{5})^{n+1}], \qquad \text{… (6)}$$

where $9+4\sqrt{5}$ is the fundamental solution of the Pellian $Y^2-5X^2=1$.

In view of the equations (2), (3), the solutions [1] of (4) are given by

$$y=\frac{1}{5}[(9+4\sqrt{5})^{n+1}(3-\sqrt{5})+(9-4\sqrt{5})^{n+1}(3+\sqrt{5})-6], \qquad \ldots(7)$$

$$x=\frac{1}{5}[(9+4\sqrt{5})^{n+1}(\sqrt{5}-1)-(9-4\sqrt{5})^{n+1}(\sqrt{5}+1)+2], \qquad \ldots.(8)$$

where $n = 0, 1, 2,\ldots..$

The values of x and y will be in integer only when $n = 2r - 1$, $r = 1, 2, 3,\ldots$

For simplicity and brevity some values of x and y are presented in the following table

Table

S. No.	Values of n	Values of x	Values of y
1	1	80	48
2	3	25632	15840
3	5	8253296	5100816
4	7	2657535552	1642447296
5	9	8557181994320	28862928880

It is noted that all the values of x and y are even. Further it is observed that the values x and y are satisfied the following recurrence relations:

(i) $x_{2r+3}-322x_{2r+1}+x_{2r-1}=-128$,

where $$x_{2r-1}=\frac{1}{5}[(9+4\sqrt{5})^{2r}(\sqrt{5}-1)-(9-4\sqrt{5})^{2r}(\sqrt{5}+1)+2], \; r = 1, 2, 3\ldots.$$

(ii) $y_{2r+3}-322\,y_{2r+1}+y_{2r-1}=384$,

where $$y_{2r-1}=\frac{1}{5}[(9+4\sqrt{5})^{2r}(3-\sqrt{5})+(9-4\sqrt{5})^{2r}(3+\sqrt{5})-6], \; \text{r} = 1, 2, 3\ldots.$$

Proof of (i)

Let $$x_{2r-1}=\frac{1}{5}[(9+4\sqrt{5})^{2r}(\sqrt{5}-1)-(9-4\sqrt{5})^{2r}(\sqrt{5}+1)+2] \qquad \ldots(9)$$

The above equation (9) may be written as

$$[5x_{2r-1}-2]=(9+4\sqrt{5})^{2r}(\sqrt{5}-1)-(9-4\sqrt{5})^{2r}(\sqrt{5}+1) \qquad \ldots(10)$$

If $A=9+4\sqrt{5}$ and $B=\sqrt{5}+1$, then we have $9-4\sqrt{5}=A-8\sqrt{5}$, and $\sqrt{5}-1=B-2$

Then the equation (10) becomes

$$[5x_{2r-1}-2]=A^{2r}(B-2)-(A-8\sqrt{5})^{2r}B \qquad \ldots(11)$$

Replacing r by $r+1$, $r+2$ successively in (11), we get

$$[5x_{2r+1}-2]=A^{2r+2}(B-2)-(A-8\sqrt{5})^{2r+2}B, \qquad \ldots(12)$$

$$[5x_{2r+3}-2]=A^{2r+4}(B-2)-(A-8\sqrt{5})^{2r+4}B. \qquad \ldots(13)$$

Multiplying the equation (11) by A^2 and then subtracting from the equation (12), we get

$$[5x_{2r+1}-2]-[5x_{2r-1}-2]A^2 = B(A-8\sqrt{5})^{2r}[A^2-(A-8\sqrt{5})^2] \qquad \dots (14)$$

Multiplying the equation (12) by A^2 and then subtracting from the equation (13), we get

$$[5x_{2r+3}-2]-[5x_{2r+1}-2]A^2 = B(A-8\sqrt{5})^{2r+2}[A^2-(A-8\sqrt{5})^2] \qquad \dots (15)$$

Multiplying the equation (14) by $(A-8\sqrt{5})^2$ and then subtracting from the equation (15), we get

$$[5x_{2r+3}-2]-[5x_{2r+1}-2][A^2+(A-8\sqrt{5})^2]+[5x_{2r-1}-2]A^2(A-8\sqrt{5})^2 = 0$$

$$[5x_{2r+3}-2]-[5x_{2r+1}-2][(9+4\sqrt{5})^2+(9-4\sqrt{5})^2]+[5x_{2r-1}-2](9+4\sqrt{5})^2(9-4\sqrt{5})^2 = 0,$$

where $A = 9+4\sqrt{5}.$

On simplifying we get,

$$[5x_{2r+3}-2]-322[5x_{2r+1}-2]+[5x_{2r-1}-2]=0$$

$$5x_{2r+3}-1610\,x_{2r+1}+5x_{2r-1} = -640$$

On dividing by 5, we get

$$x_{2r+3}+322\,x_{2r+1}+x_{2r-1} = -128 \qquad \dots (16)$$

This is the recurrence relation satisfied by the values of x.

It is observed that the values of x : (80, 25632, 8253296); (25632, 8253296, 2657535552); and (8253296, 2657535552, 855718194320) are satisfied by the above equation (16).

Proof of (ii)

Let $$y_{2r-1} = \frac{1}{5}[(9+4\sqrt{5})^{2r}(3-\sqrt{5})+(9-4\sqrt{5})^{2r}(3+\sqrt{5})-6] \qquad \dots (17)$$

The above equation (17) may be written as

$$[5y_{2r-1}+6] = (9+4\sqrt{5})^{2r}(3-\sqrt{5})+(9-4\sqrt{5})^{2r}(3+\sqrt{5}) \qquad \dots (18)$$

If $A = 9+4\sqrt{5}$ and $B = 3+\sqrt{5}$, then we have $9-4\sqrt{5} = A-8\sqrt{5}$, and $3-\sqrt{5} = B-2\sqrt{5}$

Then the equation (18) becomes

$$[5y_{2r-1}+6] = A^{2r}(B-2\sqrt{5})+(A-8\sqrt{5})^{2r}B \qquad \dots (19)$$

Replacing r by $r+1$, $r+2$ successively in (19), we get

$$[5y_{2r+1}+6] = A^{2r+2}(B-2\sqrt{5})+(A-8\sqrt{5})^{2r+2}B, \qquad \dots (20)$$

$$[5y_{2r+3}+6] = A^{2r+4}(B-2\sqrt{5})+(A-8\sqrt{5})^{2r+4}B\,. \qquad \dots (21)$$

Multiplying the equation (19) by A^2 and then subtracting from the (20), we get

$$[5y_{2r+1}+6]-[5y_{2r-1}+6]A^2 = B(A-8\sqrt{5})^{2r}[(A-8\sqrt{5})^2-A^2] \qquad \dots (22)$$

Multiplying the equation (20) by A^2 and then subtracting from the (21), we get

$$[5y_{2r+3}+6]-[5y_{2r+1}+62]A^2 = B(A-8\sqrt{5})^{2r+2}[(A-8\sqrt{5})^2-A^2] \qquad \dots (23)$$

Multiplying the equation (22) by $(A-8\sqrt{5})^2$ and then subtracting from the equation (23), we get

$$[5y_{2r+3}+6]-[5y_{2r+1}+6][A^2+(A-8\sqrt{5})^2]+[5y_{2r-1}+6]A^2(A-8\sqrt{5})^2=0$$

$$[5y_{2r+3}+6]-[5y_{2r+1}+6][(9+4\sqrt{5})^2+(9-4\sqrt{5})^2]+[5y_{2r-1}+6](9+4\sqrt{5})^2(9-4\sqrt{5})^2=0$$

On simplifying we get,

$$[5y_{2r+3}+6]-322[5y_{2r+1}+6]+[5y_{2r-1}+6]=0$$

$$5y_{2r+3}-1610\,y_{2r+1}+5y_{2r-1}=1920$$

On dividing by 5, we get

$$y_{2r+3}-322\,y_{2r+1}+y_{2r-1}=384. \qquad \ldots(24)$$

This is the recurrence relation satisfied by the values of y.

It is observed that the values of y : (48, 15840, 5100816); (15840, 5100816, 1642447296); (5100816, 1642447296, 52886292880) are satisfied by the above equation (24).

*C*ONCLUSION

One may search for other integral solutions of (7), and (8).

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