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WEAKLY αĝ-NORMAL SPACES

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The aim of this paper is to introduce a new class of normal spaces called wαĝ - normal spaces by using wαĝ-open sets due to Navalagi and. Patil [**4**] and obtained several properties of such a space. Moreover, we obtain some new characterizations and preservation theorems of wαĝ-normal spaces.

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INTRODUCTION

The aim of this paper is to introduce a new class of normal spaces called wαĝ-normal spaces by using wαĝ-open sets and obtained several properties of such a space. Navalagi and. Patil [4] introduced the concept of wαĝ-closed sets and discuss some of their basic properties. Recently, Devamanoharan [1] introduced the concept of ρ-regular and ρ-normal spaces by using ρ-closed sets and obtained several properties of such spaces. Throughout this paper, (X, τ), (Y, σ) spaces always mean topological spaces X, Y respectively on which no separation axioms are assumed unless explicitly stated.

PRELIMINARIES

2.1. Definition. A subset A of a topological space X is called

1. **α-closed** [6] if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$.
2. ***g*α-closed** [3] if $\alpha\text{-cl}(A) \subseteq U$ whenever $A \subseteq U$, and U is an α-open in X.
3. **α*gs*-closed** [8] if $\alpha\text{-cl}(U) \subseteq U$ whenever $A \subset U$ and U is semi-open in X.
4. **α*ĝ*-closed** [5] if $\text{cl}(U) \subseteq U$ whenever $A \subseteq U$ and U is α*gs*-open in X.
5. ***w*α*ĝ*-closed** [4] if $\text{cl}(\text{int}(A)) \subseteq U$ whenever $A \subseteq U$ and U is α*gs*-open in X.
6. ***rg*β-closed** [7] if $\beta\text{-cl}\, A \subseteq U$ whenever $A \subseteq U$ and U is regular open in X.
7. ***gsp*-closed** [2] if $\text{sp-cl}(U) \subset U$ whenever $A \subset U$ and U is open in X.

The complement of α-closed (resp. *g*α-closed, α*gs*-closed, α$\hat{g}$-closed, *w*α$\hat{g}$-closed, *rg*β-closed, *gsp*-closed) set is said to be **α-open** (resp. ***g*α-open**, **α*gs*-open α$\hat{g}$-open *w*α$\hat{g}$-open, *rg*β-closed**, ***gsp*-open**) set. The intersection of all *w*α$\hat{g}$-closed subset of *X* containing *A* is called the ***w*α$\hat{g}$-closure of *A*** and is denoted by ***w*α$\hat{g}$-cl(*A*)**. The union of all *w*α$\hat{g}$-open sets contained in *A* is called ***w*α$\hat{g}$-interior of *A*** and is denoted by ***w*α$\hat{g}$-int(*A*)**. The family of *w*α$\hat{g}$-open (resp. *w*α$\hat{g}$-closed) sets of a space *X* is denoted by ***W*α$\hat{G}$*O*(*X*)** (resp. ***W*α$\hat{G}$*C*(*X*)**).

2.2. Remark . Every α-closed (resp. α-open) set is *w*α$\hat{g}$-closed (resp. *w*α$\hat{g}$-open) set.

Definitions stated above, we have the following diagram:

closed ⇒ α-closed ⇒ α$\hat{g}$-closed ⇒ *g*α-closed ⇒ *w*α$\hat{g}$- closed ⇒ *gsp*-closed ⇒ *rg*β-closed

However the converses of the above are not true may be seen by the following examples.

2.3. Example. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, X\}$. Then $A = \{b\}$ is *w*α$\hat{g}$-closed set as well as *gsp*-closed set but not α$\hat{g}$-closed set in *X*.

2.4. Example. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Then the sets $\{a\}$, $\{b\}$ is *gsp*-closed set but not *w*α$\hat{g}$-closed set in *X*.

2.5. Example. Let $X = \{a, b, c\}$, $\tau = \{X, \varphi, \{a\}, \{b\}, \{a, b\}\}$. Then the sets $\{a, b\}$ is *rg*β-closed but not *gsp*-closed

2.6. Example. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Then the set $\{a\}$ is *w*α$\hat{g}$-closed in *X* but not *g*α-closed set.

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3.1. Definition. A topological space *X* is said to be **normal** (resp. ***w*α$\hat{g}$-normal**) if for every pair of disjoint closed sets *A* and *B*, there exist open (resp. *w*α$\hat{g}$-open) sets *U* and *V* such that $A \subset U$ and $B \subset V$.

3.2. Example. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then $A = \{a\}$ and $B = \phi$ are disjoint closed sets, there exist disjoint open sets $U = \{a, c, d\}$ and $V = \{b\}$ such that $A \subset U$ and $B \subset V$. Hence *X* is normal as well as *w*α$\hat{g}$-normal because every open set is *w*α$\hat{g}$-open set.

3.3. Remark. By the definitions and examples stated above, we have the following diagram :

normal ⇒ α-normal ⇒ *w*α$\hat{g}$-normal.

3.5. Definition. A function $f : X \to Y$ is said to be

1. *w*α$\hat{g}$-closed if $f(F)$ is *w*α$\hat{g}$-closed in *Y* for every closed set *F* of *X*.

2. almost *w*α$\hat{g}$-closed if for every regular closed set *F* of $X, f(F)$ is *w*α$\hat{g}$-closed in *Y*.

3. almost *gsp*-closed if for every regular closed set *F* of $X, f(F)$ is *gsp*-closed in *Y*.

4. almost *rg*β-closed if for every regular closed set *F* of $X, f(F)$ is *rg*β-closed in *Y*.

We have the following diagram for properties of functions:

*w*α$\hat{g}$-closed ⇒ almost *w*α$\hat{g}$-closed ⇒ almost *gsp*-closed ⇒ almost *rg*β-closed

3.6. Lemma. A subset *A* of a topological space *X* is *w*α$\hat{g}$-open iff $F \subseteq \text{int}(\text{cl}(A))$ whenever *F* is α*gs*-closed and $F \subseteq A$.

3.7. Theorem. A surjection $f : X \to Y$ is a almost $w\alpha\hat{g}$-closed if and only if for each subsets S of Y and each $U \in RO(X)$ containing $f^{-1}(S)$ there exists a $w\alpha\hat{g}$-open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Proof. Necessity. Suppose that f is almost $w\alpha\hat{g}$-closed. Let S be a subset of Y and $U \in RO(X)$ containing $f^{-1}(S)$. If $V = Y - f(X - U)$, then V is a $w\alpha\hat{g}$-open set of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

Sufficiency. Let F be any regular closed set of X. Then $f^{-1}(Y - f(F)) \subset X - F$ and $X - F \in RO(X)$. There exists a $w\alpha\hat{g}$-open set V of Y such that $Y - f(F) \subset V$ and $f^{-1}(V) \subset X - F$. Therefore, we have $f(F) \supset Y - V$ and $F \subset X - f^{-1}(V) \subset f^{-1}(Y - V)$. Hence we obtain $f(F) = Y - V$ and $f(F)$ is $w\alpha\hat{g}$-closed in Y. This shows that f is almost $w\alpha\hat{g}$-closed.

3.8. Theorem. For a topological space X, the following are equivalent :

(a) X is $w\alpha\hat{g}$-normal.

(b) For any disjoint closed sets A and B, there exist disjoint *gsp*-open sets U and V such that $A \subset U$ and $B \subset V$.

(c) For any disjoint closed sets A and B, there exist disjoint $rg\beta$-open sets U and V such that $A \subset U$ and $B \subset V$.

(d) For any closed sets A and any open set B containing A, there exists *gsp*-open set U of X such that $A \subset U \subset w\alpha\hat{g}\text{-cl}(U) \subset B$.

(e) For any closed sets A and any open set B containing A, there exists $rg\beta$-open set U of X such that $A \subset U \subset w\alpha\hat{g}\text{-cl}(U) \subset B$.

Proof. (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c) , (d) $\Rightarrow$ (e), (c) $\Rightarrow$ (d) and (e) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). Let X be a $w\alpha\hat{g}$-normal. Let A, B be disjoint closed sets in X. By assumption, there exist disjoint $w\alpha\hat{g}$-open sets U and V such that $A \subset U$ and $B \subset V$. Since every $w\alpha\hat{g}$-open set is *gsp*-open set, U, V are *gsp*-open sets such that $A \subset U$ and $B \subset V$.

(b) $\Rightarrow$ (c). Let A, B be disjoint closed sets. By assumption, there exist *gsp*-open sets U and V such that $A \subset U$ and $B \subset V$. Since every *gsp*-open set is $rg\beta$-open set, U, V are $rg\beta$-open sets such that $A \subset U$ and $B \subset V$.

(d) $\Rightarrow$ (e). Let A be any closed set and B be any open set containing A. By assumption, there exists *gsp*-open set U of X such that $A \subset U \subset w\alpha\hat{g}\text{-cl}(U) \subset B$. Since every *gsp*-open set is $rg\beta$-open set, there exists $rg\beta$-open set U of X such that $A \subset U \subset w\alpha\hat{g}\text{-cl}(U) \subset B$.

(c) $\Rightarrow$ (d). Let A be any closed set and B be any open set containing A. By assumption, there exist $rg\beta$-open set U and W such that $A \subset U$ and $X - B \subset W$. By **Lemma 3.6** we get, $X - B \subset w\alpha\hat{g}\text{-int}(W)$ and $w\alpha\hat{g}\text{-cl}(U) \cap w\alpha\hat{g}\text{-int}(W) = \phi$.

Hence, $A \subset U \subset w\alpha\hat{g}\text{-cl}(U) \subset X - w\alpha\hat{g}\text{-int}(W) \subset B$.

(e) $\Rightarrow$ (a). If A and B be any two disjoint closed sets of X. Then $A \subset X - B$ and $X - B$ is a open. By assumption there exist $rg\beta$-open set G of X such that $A \subset G \subset w\alpha\hat{g}\text{-cl}(G) \subset X - B$. Put $U = w\alpha\hat{g}\text{-int}(G)$, $V = X - w\alpha\hat{g}\text{-cl}(G)$. Then U and V are disjoint $w\alpha\hat{g}$-open sets of X such that $A \subset U$ and $B \subset V$. Therefore, X is $w\alpha\hat{g}$-normal.

3.9. Theorem. If $f : X \to Y$ is a continuous almost $w\alpha\hat{g}$-closed surjection and X is normal space, then Y is $w\alpha\hat{g}$-normal.

Proof. Let A and B be any two disjoint closed sets of Y. Then f^{-1} (A) and f^{-1} (B) are disjoint closed set of X. Since X is normal, there exists disjoint open set U and V such that f^{-1} (A) $\subset U$ and f^{-1} (B) $\subset V$. Let G = int(cl(U)) and H = int(cl$V$)). Then G and H are disjoint regularly open sets of X such that $f^{-1}(A) \subset G$ and $f^{-1}(B) \subset H$. Set K and L are $w\alpha\hat{g}$-open sets of Y such that, $A \subset K$ and $B \subset L$, f^{-1} (K) $\subset G$ and f^{-1} (L) $\subset H$. By **Theorem 3.7**, there exist disjoint $w\alpha\hat{g}$-open sets K and L of Y such that, $A \subset K$ and $B \subset L$, f^{-1} (K) $\subset G$ and f^{-1} (L) $\subset H$. Since G and H are disjoint, so are K and L. It follows from **Theorem 3.7**, that Y is $w\alpha\hat{g}$-normal.

3.10. Corollary. If $f : X \to Y$ is a continuous, $w\alpha\hat{g}$-closed surjection and X is normal space, then Y is $w\alpha\hat{g}$-normal.

Proof. Easy to verify.

3.11. Corollary. If $f : X \to Y$ is a continuous almost *gsp*-closed surjection and X is a normal space, then Y is $w\alpha\hat{g}$-normal.

Proof. Easy to verify.

3.12 Corollary. If $f : X \to Y$ is a continuous almost *rg*β-closed surjection and X is a normal space, then Y is $w\alpha\hat{g}$-normal.

Proof. Easy to verify.

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