Acta Ciencia Indica, Vol. XL M, No. 1, 65 (2014)

INVERSE AND DISJOINT NEIGHBOURHOOD CONNECTED DOMINATING SETS IN GRAPHS

V.R. KULLI

Department of Mathematics, Gulbarga University, Gulbarga 585 106, India

RECEIVED : 16 July, 2013

Let $G = (V, E)$ be a connected graph. Let D be a minimum neighbourhood connected dominating set of G. If $V–D$ contains a neighbourhood connected dominating set D' of G, then D' is called an *inverse neighbourhood connected dominating set* with respect to D. The *inverse neighbourhood connected domination number* $\gamma_{nc}^{-1}(G)$ of G is the minimum cardinality of a neighbourhood connected dominating set of G. The *disjoint neighbourhood connected domination number* $\gamma_{nc}\gamma_{nc}(G)$ of a graph G is the minimum cardinality of the union of two disjoint neighbourhood connected dominating sets in G. In this paper, we initiate a study of these new parameters.

KEYWORDS: Inverse neighbourhood connected domination number, disjoint neighbourhood connected domination number.

Mathematics Subject Classification: 05*C*

INTRODUCTION

All graphs considered here are finite, undirected and connected without loops or multiple edges. Any undefined term in this paper may be found in Harary [2].

For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V : uv \in E\}$ and the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighbourhood $N(S)$ of S is defined by $N(S) = \bigcup_{v \in S} N(v)$, for all $v \in S$ and the closed neighbourhood of S is $N[S] = N(S) \cup S$.

A set D of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex in $V – D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A recent survey on $\gamma(G)$ can be found in Kulli [3].

A dominating set D of a connected graph G is called a neighborhood connected dominating set (*ncd*-set) if the induced subgraph $\langle N(D) \rangle$ is connected. The neighborhood connected domination number $\gamma_{nc}(G)$ of G is the minimum cardinality of a ncd-set of G, (see [1]).

The first paper on the inverse domination number was published by Kulli and Sigarkanti [14] and is studied by several graph theorists in the world, for example, in [2, 3, 5, 6, 7, 11, 13, 17, 18].

The concept of inverse domination is as follows :

Let D be a minimum dominating set of G. If $V - D$ contains a dominating set D' of G, then D' is called an inverse dominating set of G with respect to D. The inverse domination number $\gamma^{-1}(G)$ of G is the minimum cardinality of an inverse dominating set of G.

In this paper, we introduce the concept of inverse neighborhood connected domination as follows:

Let $D \subseteq V$ be a minimum neighborhood connected dominating set of a connected graph $G = (V, E)$. If $V - D$ contains an ncd-set D' of G, then D' is called an inverse neighborhood connected dominating set (incd-set) with respect to D. The inverse neighborhood connected domination number $\gamma_{nc}^{-1}(G)$ of G is the minimum cardinality of an incd-set of G.

The upper inverse neighborhood connected domination number $\Gamma_{nc}^{-1}(G)$ of G is the maximum cardinality of an incd-set of G.

For example, we consider the graph C_6 in Figure 1. The minimum neighborhood connected dominating sets of C_6 are {1, 2, 4}, {2, 3, 5} {3, 4, 6}, {4, 5, 1}, {5, 6, 2} {6, 1, 3} and the corresponding inverse neighborhood connected dominating sets are {3, 5, 6}, {4, 6, 1}, {5, 1, 2}, {6, 2, 3}, {1, 3, 4}, {2, 4, 5} respectively. Therefore $\gamma_{nc}(C_6) = 3$, $\gamma_{nc}^{-1}(C_6) = 3$ and $\Gamma_{nc}^{-1}(C_6) = 3$.

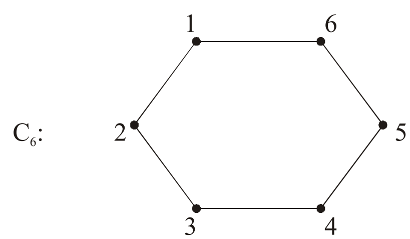

Figure 1

A dominating set D of a graph G is a split dominating set if the induced subgraph $\langle V - D \rangle$ is disconnected. The split domination number $\gamma_s(G)$ of G is the minimum cardinality of a split dominating set of G. This concept was introduced by Kulli and Janakiram in [5].

The connectivity $\kappa(G)$ of G is the minimum number of vertices whose removal results in a trivial of disconnected graph.

A γ_{nc}^{-1}-set is a minimum inverse neighborhood connected dominating set. Let $G = (V, E)$ be a graph with $|V| = p$ and $|E| = q$. Let $\lceil x \rceil (\lfloor x \rfloor)$ denote the least (greatest) integer greater (less) than or equal to x.

An application of inverse domination is found in a computer network. In the event that there is a need for all nodes in a system to have direct access to needed resources (for example, large database) a dominating set furnishes such a configuration. If a second important resource is needed, then a separate disjoint dominating set provides duplication in case the first is corrupted in some way. So we require the inverse domination number of a graph.

PRELIMINARY RESULTS

The following will be useful in the proof of our results.

Theorem A [1]. For a path P_p, $\gamma_{nc}\left(P_p\right)=\left\lceil \frac{p}{2} \right\rceil$.

Theorem B [1]. For any graph C_p with $p \geq 3$ vertices,

$$\gamma_{nc}\left(C_p\right)=\left\lfloor \frac{p}{2} \right\rfloor, \quad \text{if } p \equiv 3 \pmod 4$$

$$=\left\lceil \frac{p}{2} \right\rceil, \quad \text{otherwise.}$$

Theorem C [5]. Let G be a graph with a γ_s-set and an endvertex. Then

$$\gamma(G) = \gamma_s(G).$$

Theorem D [5]. If G has a γ_s-set, then

$$\kappa(G) \leq \gamma_s(G).$$

RESULTS

Remark A. Not all graphs have an inverse neighborhood connected domination number. For example, the cycle C_5 has no inverse neighborhood connected domination number.

Proposition 1. If P_{2n} is a path, then $\gamma_{nc}^{-1}(P_{2n}) = n$.

Proposition 2. If C_n is a cycle with $n \geq 3$ vertices and $n = 3 \pmod 4$, then

$$\gamma_{nc}^{-1}\left(C_n\right)=\left\lfloor \frac{n}{2} \right\rfloor.$$

Proposition 3. If C_{2n+2} is a cycle with $n \geq 1$, then

$$\gamma_{nc}^{-1}\left(C_{2n+2}\right)=n+1.$$

Theorem 4. If a γ_{nc}^{-1}-set exists in a connected graph G, then

$$\gamma_{nc}(G) \leq \gamma_{nc}^{-1}(G) \qquad \ldots (1)$$

and this bound is sharp.

Proof: Clearly every inverse neighborhood connected domination number is a neighborhood connected domination number of G. Thus (1) holds.

The paths P_{2n}, $n \geq 1$ and cycles C_{2n}, $n \geq 2$ achieve this bound.

Theorem 5. If a γ_{nc}^{-1}-set exists in a graph G, then

$$\gamma_{nc}(G) + \gamma_{nc}^{-1}(G) \leq p$$

and this bound is sharp.

Proof: The proof follows from the definition of $\gamma_{nc}^{-1}(G)$.

The paths P_{2n}, $n \geq 1$ and cycles C_{2n}, $n \geq 2$ achieve this bound.

We obtain a relation between $\gamma(G)$ and $\gamma_{nc}(G)$.

Theorem 6. For any connected graph G with a γ_{nt}^{-1}-set,

$$\gamma(G) + \gamma_{nc}^{-1}(G) \leq p \qquad \ldots (2)$$

and this bound is sharp.

Proof: By definition, $\gamma(G) \leq \gamma_{nc}(G)$. By Theorem 5, $\gamma_{nc}(G) + \gamma_{nc}^{-1}(G) \leq p$. Thus (2) holds. The path P_4 and the path C_4 achieve this bound.

Proposition 7. If a γ_{nc}^{-1}-set exists in a graph G, then

$$\gamma(G) \leq \gamma_{nc}^{-1}(G)$$

and this bound is sharp.

The path P_4 and the cycle C_4 achieve this bound.

Theorem 8. For any connected graph G of order $p \geq 3$, with a γ_{nc}-set and an endvertex,

$$\gamma_s(G) \leq \gamma_{nc}(G) \qquad \ldots (3)$$

and this bound is sharp.

Proof: By definition, $\gamma(G) \leq \gamma_{nc}(G)$ and from Theorem C, $\gamma(G) = \gamma_s(G)$. Thus (3) holds.

The path P_4 achieves this bound.

Corollary 9. For any tree T with $p \geq 3$ vertices,

$$\gamma_s(T) \leq \gamma_{nc}(T).$$

Theorem 10. Let G be a connected graph with an endvertex. If a γ_{nc}^{-1}-set exists in G, then

$$\gamma_s(G) \leq \gamma_{nc}^{-1}(G)$$

and this bound is sharp.

Proof : The inequality follows from Theorem 7 and Theorem 8.

The path P_4 achieves this bound.

Corollary 11. For any tree T with $p \geq 3$ with a γ_{nc}^{-1} - set,

$$\gamma_s(T) \leq \gamma_{nc}^{-1}(T)$$

and this bound is sharp.

The path P_4 achieves this bound.

Now we obtain a relation between $\kappa(G)$ and $\gamma_{nc}^{-1}(G)$.

Theorem 12. If a γ_{nc}^{-1}-set exists, then $\kappa(G) \leq \gamma_{nc}^{-1}(G)$.

Proof : The proof follows from Theorem D [5] and Theorem 10.

The Kulli-Sigarkanti conjecture and the concept of the inverse domination number inspired Hedetniemi S.M., Hedetniemi S.T., Laskar, Markus and Slater [5] to introduce disjoint domination number and is studied, for example, in [4, 10, 15, 16].

The inverse neighborhood connected domination number inspired us to introduce the following concept.

The disjoint neighborhood connected domination number $\gamma_{nc}\gamma_{nc}(G)$ of a graph G is defined as follows: $\gamma_{nc}\gamma_{nc}(G) = \min\{|D_1| + |D_2|;\ D_1, D_2$ are disjoint neighborhood connected dominating sets of $G\}$. We say that two disjoint neighborhood connected dominating sets, whose union has cardinality $\gamma_{nc}\gamma_{nc}(G)$, is a $\gamma_{nc}\gamma_{nc}$-pair of G.

Note that not all graphs have disjoint neighborhood connected domination number. For example, the path P_5 does not have two disjoint neighborhood connected dominating sets.

Theorem 13. If a graph G has a γ_{nc}^{-1}-set, then

$$2\gamma_{nc}(G) \le \gamma_{nc}\gamma_{nc}(G) \le \gamma_{nc}(G) + \gamma_{nc}^{-1}(G) \le p.$$

We say that a graph G is called $\gamma_{nc}\gamma_{nc}$-minimum if it has two disjoint γ_{nc}-sets, that is $\gamma_{nc}\gamma_{nc}(G) = 2\gamma_{nc}(G)$. Similarly a graph G is called $\gamma_{nc}\gamma_{nc}$-maximum if $\gamma_{nc}\gamma_{nc}(G) = p$.

The disjoint domination number $\gamma\gamma(G)$ of a graph G is the minimum cardinality of the union of two disjoint dominating sets in G, [5].

When the disjoint neighborhood connected domination number exists, the following inequality holds.

Proposition 14. For any connected graph G having two disjoint neighborhood connected dominating sets,

$$\gamma\gamma(G) \le \gamma_{nc}\gamma_{nc}(G).$$

The cycle C_4 and the path P_4 achieve this bound.

The exact values of $\gamma_{nc}\gamma_{nc}(G)$ for some standard graphs are given below.

Proposition 15. For any path P_{2n}, $n \ge 1$,

$$\gamma_{nc}\gamma_{nc}(P_{2n}) = 2n.$$

Proof : This follows from Theorem A and Proposition 1.

Proposition 16. For any path P_{2n+2}, $n \ge 1$,

$$\gamma_{nc}\gamma_{nc}(P_{2n+2}) = 2n + 2.$$

Proof : This follows from Theorem B and Proposition 3.

Proposition 17. If C_n is a cycle with $n \ge 3$ and $n = 3 \pmod 4$, then

$$\gamma_{nc}\gamma_{nc}(C_n) = n - 1.$$

Proof : This follows from Theorem B and Proposition 2.

The cycles C_n, $n \ge 3$ and $n = 3 \pmod 4$, the cycles C_{2n+2} $n \ge 1$ and the paths P_{2n}, $n \ge 1$ are $\gamma_{nc}\gamma_{nc}$-minimum.

The cycles $C_{2n+2,}$ $n \ge 1$ and the paths P_{2n}, $n \ge 1$ are $\gamma_{nc}\gamma_{nc}$-maximum.

The cycles $C_{2n,}$ $n\ge3$ and $n=3 \pmod 4$ are $\gamma_{nc}\gamma_{nc}$-maximum.

Some open problems

In this paper we have introduced a new type of inverse domination, namely, inverse neighborhood connected domination. Also we introduced disjoint neighborhood connected domination. Many questions are suggested by this research, among them are the following.

Problem 1. Characterize graphs for which $\gamma_{nc}(G) = \gamma_{nc}^{-1}(G)$.

Problem 2. Characterize graphs for which $\gamma_{nc}(G) + \gamma_{nc}^{-1}(G) = p$.

Problem 3. Characterize graphs for which $\gamma(G) + \gamma_{nc}^{-1}(G) = p$.

Problem 4. Characterize graphs for which $\gamma(G) = \gamma_{nc}^{-1}(G)$.

Problem 5. Characterize graphs for which $\gamma\gamma(G) = \gamma_{nc}\gamma_{nc}(G)$.

Problem 6. Characterize the class of $\gamma_{nc}\gamma_{nc}$-minimum graphs.

Problem 7. Characterize the class of $\gamma_{nc}\gamma_{nc}$-maximum graphs.

Acknowledgement

The author thanks to referee for his/her suggestions.

References

1. Arumugam, S. and Sivagnanam, C., Neighborhood connected domination in graphs, *J.Combin. Math. Combin. Comput.*, **75**, 239-249 (2010).
2. Domke, G.S., Dunbar, J.E. and Markus, L.R., The inverse domination number of a graph, *Ars Combin.*, **72**, 149-160 (2004).
3. Frendrup, A., Henning, M.A., Randerath, B. and Vestergaard, P.D., On a conjecture about inverse domination in graphs, *Ars. Combinatoria*, **95A**, 103-111 (2010).
4. Harary, *Graph Theory*, Addison Wesley, Reading Mass (1969).
5. Hedetniemi, S.M., Hedetniemi, S.T., Laskar, R.C., Markus, L. and Slater, P.J., Disjoint dominating sets in graphs, Discrete Mathematics, *Ram. Math. Soc. Lect. Notes*, Series **7**, Ram. Math. Soc. Mysore, 87-100 (2008).
6. Johnson Jr., P.D., Prier, D.R. and Walsh, M., On a problem of Domke, Haynes, Hedetniemi and Markus concerning the inverse domination number, *AKCE J. Graphs Combin.*, **7**, No. **2**, 217-222 (2010).
7. Johnson, P.D. and Walsh, M., Fractional inverse and inverse fractional domination in graphs, *Ars. Combin.*, **87**, 13-21 (2008).
8. Kulli, V.R., *Theory of Domination in Graphs,* Vishwa International Publications, Gulbarga, India (2010).
9. Kulli, V.R., The disjoint covering number of a graph, *International J. of Math. Sci. & Engg. Appls.,* **7**, No. **V**, 135-141 September (2013).
10. Kulli, V.R., Inverse and disjoint neighborhood total dominating sets in graphs, preprint.
11. Kulli, V.R. and Iyer, R.R., Inverse total domination in graphs, *J. Discrete Mathematical Sciences and Cryptography*, **10(5)**, 613-620 (2007).
12. Kulli, V.R. and Janakiram, B., The split domination number of a graph, *Graph Theory Notes of New York, New York Academy of Sciences,* **XXXII**, 16-19 (1997).
13. Kulli, V.R. and Kattimani, M.B., The inverse neighborhood number of a graph, *South East Asian J. Math. & Math. Sci.,* **6(3)**, 23-28 (2008).
14. Kulli, V.R. and Sigarkanti, S.C., Inverse domination in graphs, *Nat. Acad. Sci. Lett.,* **14**, 473-475 (1991).
15. Löwenstein, C., In the complement of a dominating set, *Dissertation, Ilmenau*, Germany (2010).
16. Löwenstein, C. and Rautenbach, D., Pairs of disjoint dominating sets and minimum degree of graphs, *Graphs and Combinatorics,* **26**, 407-424 (2010).
17. Prier, D., The inverse domination problem, DI-Pathological graphs and fractional analogues, *Ph.D. in Mathematics*, Auburn University, Auburn, USA (2010).
18. Chelvam. T. Tamizh and Asir, T., Graphs with constant sum of domination and inverse domination, *Inter. J. Combinatorics*, 2012 (2012).

❑