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ONMIXED TRILATERAL GENERATING RELATIONS FOR BIORTHOGONAL POLYNOMIALS SUGGESTED BY LAGUERRE POLYNOMIALS

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In this note, we have obtained some novel mixed trilateral generating functions involving Konhauser biorthogonal polynomials $Y_n^\alpha(x;k)$ by group theoretic method. As special cases, we have obtained the corresponding results on generalised Laguerre polynomials.

KEYWORDS : *Laguerre polynomials, biorthogonal polynomials, generating functions.*

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Introduction

An explicit representation for the Konhauser biorthogonal polynomials, $Y_n^\alpha(x;k)$as introduced by Knohauser [1] was given by Carlitz [2] in the following form:

$$Y_n^\alpha(x;k) = \frac{1}{n!}\sum_{i=0}^{n}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_n,$$

where$(a)_n$ is the pochhammer symbol [3; p-273]

In a recent paper [6], the present authors have proved the following theorem on bilateral generating relations involving biorthogonal polynomials, $Y_n^\alpha(x;k)$.

Theorem 1. If there existsa unilateral generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n\, Y_n^\alpha(\mathrm{x};\mathrm{k})\, w^n, \qquad \ldots (1.1)$$

then $$(1+kw)^{\frac{(1+\alpha-k)}{k}} \exp\left(x\left[1-(1+kw)^{\frac{1}{k}}\right]\right) G\left(x(1+kw)^{\frac{1}{k}},\ wv\right)$$

$$= \sum_{n=0}^{\infty} w^n\, \sigma_n(x,v), \qquad \ldots (1.2)$$

where $$\sigma_n(x,v) = \sum_{p=0}^{n} a_p\, k^{n-p}\binom{n}{p} Y_n^{\alpha+kp-nk}(\mathrm{x};\mathrm{k})\, v^p.$$

The aim at presenting this paper is to generalise the above bilateral generating relation into mixed trilateral generating relation by the group-theoretic method. A particular cases of interest is also discussed in this paper. The main results of our investigation are stated in the form of the following theorems:

Theorem 2. If there exists a generating relation of the form

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha}(\mathrm{x};\mathrm{k})\, g_n(u)\, w^n, \qquad \dots (1.3)$$

where $g_n(u)$ is an arbitrary polynomial of degree n and left hand series have formal power series expansion, then

$$(1+kw)^{\frac{(1+\alpha-k)}{k}} \exp\left(x\left[1-(1+kw)^{\frac{1}{k}}\right]\right) G\left(x(1+kw)^{\frac{1}{k}},\ u,\ wv\right)$$

$$= \sum_{n=0}^{\infty} w^n\, \sigma_n(x,u,v), \qquad \dots (1.4)$$

where $$\sigma_n(x,u,v) = \sum_{m=0}^{n} a_m\, k^{n-m} \binom{n}{m} Y_n^{\alpha+mk-nk}(x;k)\, g_m(u)\, v^m.$$

OPERATOR AND EXTENDED FORM OF THE GROUP

At first, we seek a linear partial differential operator R of the form:

$$R = A_1(x,y,z)\frac{\partial}{\partial x} + A_2(x,y,z)\frac{\partial}{\partial y} + A_3(x,y,z)\frac{\partial}{\partial z} + A_0(x,y,z),$$

where each $A_i (i = 0,1,2,3)$ is a function of x, y and z which is independent of n, α such that

$$R[Y_n^{\alpha}(x;k)\, y^{\alpha} z^n] = c(n,\alpha)\, Y_{n+1}^{\alpha-k}(x;k)\, y^{\alpha-k} z^{n+1}, \qquad \dots (2.1)$$

where $c(n,\alpha)$ is a function of $n, \alpha$and is independent of x, y and z.

Using (2.1) and with the help of the differential recurrence relation:

$$x\frac{d}{dx}[Y_n^{\alpha}(x;k)] = k(n+1)Y_{n+1}^{\alpha-k}(x;k) + (x+k-\alpha-1)Y_n^{\alpha}(x;k) \qquad \dots (2.2)$$

We easily obtain the following linear partial differential operator:

$$R = x\, y^{-k} z\frac{\partial}{\partial x} + y^{-(k-1)} z\frac{\partial}{\partial y} - (x+k-1)y^{-k}z$$

Such that

$$R(Y_n^{\alpha}(\mathrm{x};\mathrm{k}) y^{\alpha} z^n) = k(n+1)Y_{n+1}^{\alpha-k}(x;k) y^{\alpha-k} z^{n+1}. \qquad (2.3)$$

The extended form of the group generated by R is given by

$$e^{wR} f(x,y,z) = (1+kwy^{-k}z)^{\frac{1-k}{k}} \exp\left(x - x(1+kwy^{-k}z)^{\frac{1}{k}}\right)$$

$$\times f\left(x(1+kwy^{-k}z)^{\frac{1}{k}},\ y(1+kwy^{-k}z)^{\frac{1}{k}},\ z\right), \qquad \dots (2.4)$$

where$f(x,y,z)$ is an arbitrary function and w is an arbitrary constant.

Now we proceed to prove the Theorem 2.

Proof of theorem 2

Let us consider the generating relation of the form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha}(x;k)\, g_n(u)\, w^n. \qquad \ldots (3.1)$$

Replacing $w$by wvz and multiplying both sides of (3.1) by y^{α} and finally operating e^{wR} on both sides, we get

$$e^{wR}\big(y^{\alpha}\, G(x,wvz)\big) = e^{wR}\left(\sum_{n=0}^{\infty} a_n (Y_n^{\alpha}(x;k) y^{\alpha} z^n)\, g_n(u)(wv)^n\right). \qquad \ldots (3.2)$$

Now the left member of (3.2), with the help of (2.4), reduces to

$$(1+kwy^{-k}z)^{\frac{(1+\alpha-k)}{k}} \exp\left(x - x(1+kwy^{-k}z)^{\frac{1}{k}}\right) y^{\alpha} G\left(x(1+kwy^{-k}z)^{\frac{1}{k}}, u, wvz\right) \ldots (3.3)$$

The right member of (3.2), with the help of (2.3), becomes

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{n} a_{n-p} k^p \binom{n}{p} Y_n^{\alpha-kp}(x;k) y^{\alpha-kp}(wz)^n\, g_{n-p}(u) v^{n-p}. \quad \ldots (3.4)$$

Now equating (3.3) and (3.4) and then substituting $y = z = 1$, we get

$$(1+kw)^{\frac{(1+\alpha-k)}{k}} \exp\left(x - x(1+kw)^{\frac{1}{k}}\right) G\left(x(1+kw)^{\frac{1}{k}}, u, wv\right)$$

$$= \sum_{n=0}^{\infty} w^n\, \sigma_n(x,u,v), \qquad \ldots (3.5)$$

where $$\sigma_n(x,u,v) = \sum_{p=0}^{n} a_p k^{n-p} \binom{n}{p} Y_n^{\alpha+kp-kn}(x;k)\, g_p(u) v^p.$$

This completes the proof of the theorem and does not seem to have appeared in the earlier works.

Special case : If we put $k = 1$, then $Y_n^{\alpha}(x;k)$ reduces to the generalized Laguerre polynomials, $L_n^{\alpha}(x)$. Thus putting $k = 1$ in the above theorem, we get the following theorem on generalised Laguerre polynomials.

Theorem 3. If there exists a generating relation of the form

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x)\, g_n(u)\, w^n, \qquad \ldots (3.6)$$

then $$(1+w)^{\alpha}\ \exp(-wx))\, G(x(1+w),u,wv) = \sum_{n=0}^{\infty} w^n\, \sigma_n(x,u,v), \qquad \ldots (3.7)$$

where $$\sigma_n(x,u,v) = \sum_{p=0}^{n} a_p \binom{n}{p} L_n^{(\alpha+p-n)}(x)\, g_p(u) v^p,$$

which is found derived in [4, 5].

Conclusions

1. From the above discussion, it is clear that whenever one knows a bilateral generating relation of the form (1.3, 3.6) then the corresponding trilateral generating relation can at once be written down from (1.4, 3.7). So one can get a large number of trilateral generating relations by attributing different suitable values to a_n in (1.3, 3.6).

2. Also many applications of our theorem-3 are given in [4].

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