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A NOTE ON ECCENTRIC IRRATIONAL NUMBERS

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The Problem of approximation of an irrational number 'θ' has been discussed on this paper, by confining the study upto approximation of quadratic irrationals only. Three theorems have been proved in order to search for an irrational number which defy finite number of approximation. These irrational numbers will be called eccentric irrational numbers.

KEYWORDS : Eccentric irrational number, convergent, quotients.

NOMENCLATURE

θ = irrational number

$\frac{p}{q}$ = any rational number

λ = any positive real number

$\frac{p_n}{q_n} = n^{th}$ convergent to θ

INTRODUCTION

Given any irrational number θ, there arises a natural question that how closely or with what degree of accuracy, it can be approximated by rational number. This key question has been the subject of Mathematical Research for over five decades. Hurwitz [1] was Probably the first to study about this rational approximation of irrational numbers.

The rational approximation of irrational numbers has been studied by many authors. Recently M. Abrate *et al* [2] Studied on Periodic representations and rational approximations of square root. M. B. Nathanson *et. al.* [3] worked on irrational numbers which are associated to sequences without geometric progressions. T. Z. Kalanov [4] analysed critically a problem of irrational numbers with Pythagorean theorem. T. Zachariades [5] Studied comprehensively about irrational numbers and real numbers in his work entitled "Reflective, Systematic and Analytic thinking in real numbers. M. Aigner [6] carried out a descriptive study on approximation of irrational numbers. F. Afzal *et. al* [7] tried to classify different real quadratic irrationals. M. A. Malik *et al* [8] applied modular group action on real quadratic irrationals. C. Elsner [9] obtained a series of error terms for rational approximations of irrational numbers. Icoltescu *et al* [10] developed a new type of continued fraction expansion.

N. J. Wild Berger [11] studied pell's equation without irrational numbers. A yavari *et al* [12] did a practical research on randomness of digits of binary expansion of irrational numbers. M. G. Voskoglow *et. al.* [13] and V. Bilo *et. al.* [14] studied on the complexity which arise in approximating rational and irrational numbers.

While working on some finite approximation problem we encountered some irrational numbers θ such that : given any $\lambda > 0$, the inequality.

$$\left|\theta - \frac{p}{q}\right| \le \frac{1}{\lambda q^2}; (p,q) = 1, \;\; q > 0$$

has either infinity of solutions in p/q or no solution in p/q. Solutions p/q are called approximations of θ. So the θ's in reference defy finite (> 0) number of approximations. We call them eccentric irrational numbers. For instance, it can be easily checked that $[0,(2,1)^*]$ is a eccentric irrational number where as $[0,(1,2)^*]$ is not an eccentric irrational. Note that $[0,(2,1)^*] \sim [0,(1,2)^*]$. This means that "eccentric behaviour" is not preserved under equivalence of irrational numbers. This makes it difficult and interesting to characterize the eccentric irrationals.. For the sake of convenience we confine to quadratic irrationals only and find :

(a) a necessary condition for a quadratic irrational number to be eccentric.

(b) a characterization of eccentric quadratic irrational of least period length 2.

Our findings are :-

Theorem 1. If θ is an eccentric quadratic irrational then the least period of the simple continued fraction expansion of θ must contain an even number of quotients.

Theorem 2. $[0, (a, b)^*]$ is an eccentric irrational number iff $a > b$.

Theorem 3. $\theta = [0.c_1.c_2.............c_p.(a.b)^*].c_p \ne b$ is a eccentric irrational number iff

$$"c_p < b, a < b \text{ and } M_i(\theta) < [0,(a,b)^*] + [(b,a)^*] \;\; \forall i = 1, 2, ...,\text{p}"$$

Or,

$$"c_p > b,\ a > b \text{ and } M_i(\theta) < [0,\ (a,\ b)^*] + [(b,\ a)^*] \;\; \forall i = 1, 2, ..., p"$$

where $M_i(\theta = [a_0.a_1, a_2......a_i, a_{i+1}........]) = [0, a_i, a_{i-1}......a_1] + [a_{i+1}, a_{i+2}.......]$

Ground work

We use the following three results [15, Theorem 184] on simple continued fraction quite often.

$(R.1)$ Suppose $\theta = [a_0, a_1.a_2, ..., a_n ....]$. Then $\left|\theta - \frac{P_n}{q_n}\right| = \frac{1}{M_n(\theta) q_n^2}$

where $\frac{P_n}{q_n} = [a_0, a_1, a_2, ..., a_n]$ the n^{th} convergent to θ

and $M_n(\theta) = [0,\ a_n, a_{n-1}, ..., a_1] + [a_{n+1},\ a_{n+2}, ...]$

(R.2) If θ is an irrational number, $\lambda > 2$ and $\left|\theta - \frac{p}{q}\right| \leq \frac{1}{\lambda q^2}$ with $q > 0$ and $(p, q) = 1$ then $\frac{p}{q}$ must be a convergent to θ.

(R.3) If k is even and $a_k < b_k$ then

$[a_0, a_1, a_2, ..., a_{k-1}, a_k, a_{k+1}, ...] < [a_0, a_1, a_2, ..., a_{k-1}, b_k, b_{k+1}, ...]$ & if k is odd

and $a_k < b_k$ then $[a_0, a_1, a_2, ..., a_{k-1}, a_k, a_{k+1}, ...] > [a_0, a_1, a_2, ..., a_{k-1}, b_k, b_{k+1}, ...]$

For given irrational number θ, let us introduce a constant $c(\theta)$ to mean that $\left|\theta - \frac{p}{q}\right| \leq \frac{1}{c(\theta) q^2}$ has exactly one solution in p/q with $q > 0$ and $(p, q) = 1$. Clearly $c(\theta)$ exists if and only if θ is not a eccentric irrational number.

We now state below the contraposed version of theorem 1, 2 and 3 which we shall prove in the next section.

Theorem 1$^+$: If the least period of simple continued fraction expansion of θ has an odd number of quotients then $c(\theta)$ exists.

Theorem 2$^+$: For $\theta = [0, (a, b)^*], (a \neq b)$; $c(\theta)$ exists iff $a < b$.

Theorem 3$^+$: $\theta = [0, c_1, c_2 ....c_p, (a, b)^*]$, $c_p \neq b$ is an eccentric irrational number iff.

"$c_p < b$, $a < b$ and $M_i(\theta) < [0, (a, b)^*] + [(b, a)^*]$ "$i = 1, 2, ..., p$"

Or,

"$c_p > b$, $a > b$ and $M_i(\theta) < [0, (a, b)^*] + [(b, a)^*]$ $\forall i = 1, 2, ..., p$"

where M_i $(\theta = [a_0, a_1, a_2, ...a_i, a_{i+1}....]) = [0, a_i, a_{i-1}....a_1] + [a_{i+1}. a_{i+2}, ...]$

Our proof requires a better understanding of $c(\theta)$. And hence the following lemma.

Lemma : Suppose $\theta = [a_0, a_1, a_2, ..., a_n ....]$.. If $c(\theta)$ exists then $c(\theta) > 2$ and $c(\theta) = \max_n \; M_n(\theta)$

Proof : As obtained in [1], If θ is equivalent to $[0, (1)^*]$, $c(\theta) = \frac{(3+\sqrt{5})}{2} > 2$. So if $c(\theta) \leq 2$ then for infinitely many values of k, $a_{k+1} \geq 2$

$$\Rightarrow M_k(\theta) > 2$$

$$\Rightarrow M_k(\theta) > c(\theta)$$

$$\Rightarrow \left|\theta - \frac{Pk}{qk}\right| = \frac{1}{M_k(\theta) qk^2} < \frac{1}{c(\theta) qk^2}$$

This contradicts the definition of $c(\theta)$ and hence $c(\theta) > 2$. Now (R. 2) implies that any solution of $\left|\theta - \frac{P}{q}\right| \le \frac{1}{c(\theta)q^2}$ must be a convergent to θ. So $c(\theta) = \max_n \; M_n(\theta)$.

PROOF OF THE THEOREM

Proof of theorem 1⁺ : Suppose

$$\theta = [0, b_1, b_2, ...b_p, (a_1, a_2, ...a_{2r-1})^*], (b_p \ne a_{2r-1}).$$

Consider the Sequence $M_{p+1}(\theta),\ M_{p+2}(\theta),\ .....\ M_{p+m}(\theta), ...$ We now subdivide this sequence in $4r-2$

Sub sequences : $\{s_m^{(0)}\}_{m\ge1},\ \{s_m^{(1)}\}_{m\ge1},\ \{s_m^{(2)}\}_{m\ge1}, ...\ \{s_m^{(4r\text{-}3)}\}_{m\ge1}.$

Here $S_m^j = M_{p+(4r-2)m-j}(\theta) = F_1^{(j)} + F_2^{(j)}\ \forall\, j = 0,\ 1,\ 2, ..., 4r-3$

where $F_1^{(j)} = [0,\ a_{2r-j-1 ....}, a_{2r-1 ....},\ a_1,\ (a_{2r-1, ...,} a_1)_{2m-2}, b_p, ..., b_1]$

and $F_2^{(j)} = [c_j,\ c_{j-1, ...,} c_{1,} (a_1,\ a_{2, ...}, a_{2r-1})^*]$ where $< c_j,\ c_{j-1, ...,} c_{1,} >$ is the portion of 1st j terms of the sequence $< a_{2r\,-\,1},\ a_{2r\text{-}2}, ..., a_1,\ a_{2r-1},\ a_{2r\text{-}2}, ..., a_1 >$.

Since b_p is an odd quotient of $F_1^{(0)}$, if $b_p < a_{2r-1}$ the sequence $\{S_m^{(0)}\}$ monotonically decrease. Also since b_p become alternately an even and odd quotient in $F_1^{(1)}, F_2^{(1)},$ etc.., the sequence $\{S_m^{(1)}\}$, the sequence $\{S_m^{(0)}\},\ \{S_m^{(1)}\}$ and so on, alternately increase and decrease. Similarly if $b_p > a_{2r-1}$ it can be checked that the sequence $\{S_m^{(0)}\},\ \{S_m^{(1)}\}$, etc. alternately increase and decrease. So in $(4r-2)$ sub sequences listed above $(2r-1)$ sub sequence are monotonically decreasing and the rest are monotonically increasing. Further it can be easily checked that the sequence $\{S_m^{(1)}\}$ and $\{S_m^{(i+2r\text{-}1)}\}$ where $0 \le i \le 2r-2$, One sequence monotonically increase and other monotonically decreases to the same limit. We not pick the first term of each of $(2r\text{-}1)$ decreasing sub sequences noted above.

Let they be $\lambda_0, \lambda_1, ..., \lambda_{2r\text{-}2}$. Then

$$\lambda = \max\,(\lambda_0, \lambda_1\ ....., \lambda_{2r-2})$$
$$= \max_{n\ge1}\ \{M_{n+p}(\theta)\}$$

So max $\{\lambda,\ M_1(\theta),\ M_2(\theta),\ ......,\ M_p(\theta)\ \} = \max_n\{M_n(\theta)\} = c(\theta)$

This proves 1⁺ and hence theorem 1.

Proof of theorem 2⁺ : Suppose $\theta = [0, (a, b)^*],\ (a \ne b)$.

Then $M_{2m}(\theta) = [0,\ b,\ a,\ (b,\ a)_{m-1}] + [(a,\ b)^*]$

and $M_{2m-1}(\theta) = [0,\ a,\ (b,\ a)_{m-1}] + [(b,\ a)^*]$. Clearly the sequence $\{M_{2m}\ (\theta)\}$ monotonically increases to $[(a,\ b)^*] + [(b,\ a)^*] = K$ (say) and the sequence $\{M_{2m}\ (\theta)\}$ monotonically decreases to $[0, (a, b)^*] + [(b, a)^*] = L$ (say) as $m \to \infty$. We argue via two cases:

Case 1. $a > b$. Then $K > 2$, $K > L$ and $M_2\ (\theta) > M_1(\theta)$ and hence $M_n(\theta) < K\ \forall\ n \geq 1$.

So $\left|\theta - \frac{P_n}{q_n}\right| \leq \frac{1}{Kq_n^2}$ does not hold for any $n \geq 1$. Since $K > L$. We infer that $\left|\theta - \frac{P}{q}\right| \leq \frac{1}{Kq^2}$ has not solution in $\frac{p}{q}$. But as all members of the sequence $\{M_{2m}(\theta)\}$ from some point on, exceed $K - \varepsilon,\ \ \varepsilon > 0$ however small, it follows that $\left|\theta - \frac{p}{q}\right| \leq \frac{1}{(K-\varepsilon)q^2}$ holds for infinitely many values of n or $\left|\theta - \frac{p}{q}\right| \leq \frac{1}{(K-\varepsilon)q^2}$ has infinitely many solution in $\frac{p}{q}$. Thus in this case $c(\theta)$ does not exists.

Case 2. $a < b$, Then we have $K < L$. So every term of the sequences $\{M_{2m-1}(\theta)\}$ is greater than all the terms of $\{M_{2m}(\theta)\}$. Therefore in this case $M_1(\theta) = \max\limits_{n}\ M_n(\theta)$ proving $c(\theta)$ exists.

This proves theorem 2^+ and hence theorem 2

Prove of Theorem 3^+ : Suppose $\theta = [0,\ c_1,\ c_2 ....c_p, (a,\ b)^*]$; ($a \neq b$ and $c_p \neq b$)

Let us consider the sequence $M_{p+1}(\theta),\ M_{p+2}(\theta),...........$ and its subsequences $\{M_{p+2m}(\theta)\}$ and $\{M_{p+2m-1}(\theta)\}$; $m = 1,\ 2$ .... clearly.

$$M_{p+2m-1}(\theta) \to [0,\ (b,\ a)^*] + [(a,\ b)^*] = K \text{ as } m \to \theta$$

$$M_{p+2m-1}(\theta) \to [0,\ (a,\ b)^*] + [(b,\ a)^*] = L \text{ as } m \to \infty$$

Now two cases arise : case (1) $c_p < b$ and Case (2) $c_p > b$

In case (1), the sequence $\{M_{p+2m}(\theta)\}$ monotonically decreases to K and $\{M_{p+2m-1}(\theta)\}$ monotonically increases to L. So if $a > b$, then $K > L$ and $M_{p+2}(\theta) = \max\limits_{n\geq 1}\ \{M_{p+n}(\theta)\}$ *i.e.* $\max\limits_{n} M_n(\theta) = \max\{M_1(\theta).....,M_p(\theta), M_{p+2}(\theta)\}$ proving the existence of $c(\theta)$, Otherwise if $a < b$, we have $K < L$ and $M_{P+2}(\theta) < M_{P+1}(\theta)$. So max $\{M_{P+1}(\theta),\ M_{P+2}(\theta),\ M_{P+3}(\theta),........\}$ does not exist. Further if $M_1(\theta)$, ... $M_p(\theta)$ be each less than L, obviously max $\{M_1(\theta),\ ....\ M_p(\theta),\ M_{p+1}(\theta)..........\ M_n(\theta),........\}$ does not exist disproving the existence of $c(\theta)$. But if $M_j(\theta),\ M_K(\theta),.......$ and $M_q(\theta)$ exceed L

where $\{j,\ k, ..., q\} \subseteq \{1,\ 2,\ ....p\}$ than $\max_n \{M_n(\theta)\}$ equals max $\{M_j(\theta),\ M_K(\theta), ...\ M_q(\theta)\}$ and then $c(\theta)$ exists.

Case (2) requires the exactly similar treatment. *This completes the proof of theorem.*

References

1. Prasad, A.V., Note on a theorem of Hurwitz, *Jour. Lond. Math. Soc.*, 2**3**, 169-171 (1948).
2. Abrate, M., Barbero, S., Cerruti, U., Murru, N., Periodic representations and rational approximations of square root, *Journal of Approxiamation Theory,* El sevier (2013).
3. Nathanson, M. B., Bryant, K. O., Irrational Numbers associated to sequences without geometric progressions, *Arxiv Preprint Arxiv*, **1307**, 8135 (2013).
4. Kalanov, T. Z., The Critical analysis of the Pythagorean theorem and of the problem of irrational numbers, *Basic Research Journals. Org.* (2013).
5. Zacharides, T., Christou, C., Pitta-Pantazi, D., Reflective, Systematic and analytic thinking in real numbers, *Springer* (2013).
6. Aigner, M., Approximation of Irrational Numbers, *Springer* (2013).
7. Afzal, F., Afzal, Q., Malik, M.A., A classification of the real quadratic irrational, *International Mathematical Forum* (2012).
8. Malik, M. A., Zafar, M.A., Real Quadratic irrational numbers and modular group action, *Southeast Asian Bulletin of Mathematics* (2011).
9. Elsner, C., Series of error terms for rational approximations of irrational numbers, *Journal of Integer Sequences* (2011).
10. Coltesco, I., Lascu, D., A new type of continued fraction expansion, *Arxiv Preprint Arxiv*., **1010**, 4425 (2010).
11. Berger, N. J. Wild, Pell's equation without irrational numbers, *Journal of Integer Sequences* (2010).
12. Yavari, A., A practical research on *randomness of digits of binary expansion of irrational number-communications and signal processing* (2009).
13. Voskoglow, M.G., Kosyvas, G., A study on the comprehension of irrational number quadernidi, *Ricerca in Didattica* (2011).
14. Bilo, V., Mavronicolas, M., Complexity of rational and irrational Nash equilibria, *Algorithmic Game Theory,* (2011) – Springer.
15. Hardy and Wright, *An Introduction to the theory of Numbers* (1960).

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