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DISTRIBUTIVE SUBSEMILATTICE

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Generalization of distributive ideal to convex subsemilattice is called distributive convex subsemilattice or distributive subsemilattice. This paper deals with the above concept and the three characterization theorems (Theorems 3.1, 3.2 and 3.5) for distributive subsemilattice. The relation of distributive subsemilattice with the distributive ideal, standard ideal and neutral ideal is also established. Further its relation with the congruence class is also established. It is proved that the intersection of two distributive subsemilattices is either a distributive subsemilattice or it is empty.

Introduction

Gratzer, G., and Schmidt, E.T., have introduced and studied the concept of distributive ideal, standard ideal and neutral ideal which are distributive elements, standard elements and neutral elements respectively in the ideal lattice $I(L)$ of a lattice L. Gratzer, G., has posed the problem "Generalize the concept of distributive, standard and neutral ideals to convex sublattice". This problem is solved by Natarajan, R., Fried, E., Schmidt, E.T. and Chellappa, B.

In the case of join semilattice the concept of distributive ideal, standard ideal and neutral ideal have been introduced and studied by Natarajan, R., and Vairavan, L. Now the problems posed by Gratzer, G., reduced as "Generalize the concept of distributive, standard and neutral ideals to convex subsemilattice".

In this paper the generalization of distributive ideal to convex subsemilattice is introduced and established through some characterization theorems.

Preliminaries

The necessary definition, theorems and results which are used in this paper are given in this section.

The symbols $\leq$, $\nleq$, $\vee$ and $\wedge$ will denote inclusion, non-inclusion, cup or join (least upper bound) and cap or meet (greatest lower bound) in a lattice (or semilattice) while symbols $\subseteq$, $\cup$, $\cap$, $\in$, $\notin$ and ϕ will refer to set inclusion, union, intersection, membership, non-membership and empty set. Small letters a, b, c, ... will denote elements of a lattice (or semilattice). Greek letters θ, ϕ will stand for congruence relation on lattice (or semilattice).

Definition 1.1 : A binary relation 'θ' on a lattice L is said to be a congruence relation if it satisfies the following conditions

(i) θ is reflexive :

$$x \equiv x\ (\theta), \text{ for all } x \in L$$

(ii) θ is symmetric :

$$x \equiv y\ (\theta) \Rightarrow y \equiv x\ (\theta), \text{ for all } x, y \in L$$

(iii) θ is transitive :

$$x \equiv y\ (\theta) \text{ and } y \equiv z\ (\theta) \Rightarrow x \equiv z\ (\theta), \text{ for all } x, y, z \in L$$

(iv) Substitution property:

$$x \equiv x_1\ (\theta) \text{ and } y \equiv y_1\ (\theta)$$

$$\Rightarrow x \vee y \equiv (x_1 \vee y_1)\ (\theta)$$

$$x \wedge y \equiv (x_1 \wedge y_1)\ (\theta), \text{ for all } x, y, x_1, y_1 \in L$$

Theorem 1.1. A reflexive and symmetric binary relation 'θ' on a lattice L is a congruence relation if and only if the following three properties are satisfied for all x, y, z in L.

(i) $x \equiv y\ (\theta) \Leftrightarrow x \wedge y \equiv (x \vee y)\ (\theta)$

(ii) $x \leq y \leq z$, $x \equiv y\ (\theta)$ and $y \equiv z\ (\theta) \Rightarrow x \equiv z\ (\theta)$

(iii) $x \equiv y\ (\theta)$ and $x \leq y \Rightarrow x \wedge t \equiv (y \wedge t)\ (\theta)$

and $x \vee t \equiv (y \vee t)\ (\theta)$, for all $t \in L$.

Definition 1.2. A join semilattice or semilattice is a non-empty set S with binary operation '$\vee$' defined on it and satisfies the following :

(i) Idempotent law :

$$a \vee a = a, \text{ for all } a \in S$$

(ii) Commutative law :

$$a \vee b = b \vee a, \text{ for all } a, b \in S$$

(iii) Associative law :

$$A \vee (b \vee c) = (a \vee b) \vee c, \text{ for all } a, b, c, \in S$$

Definition 1.3. A semilattice is a non-empty set S with binary relation '$\leq$' defined on it and satisfies the following:

(i) '$\leq$' is reflexive:

$$a \leq a, \text{ for all } a \in S$$

(ii) '$\leq$' is antisymmetric:

$$a \leq b \text{ and } b \leq a \Rightarrow a = b, \text{ for all } a, b \in S$$

(iii) '$\leq$' is transitive:

$$a \leq b \text{ and } b \leq c \Rightarrow a \leq c, \text{ for all } a, b, c, \in S$$

(iv) any two elements in S have a least upper bound.

RESULT

Two definitions for semilattice S are equivalent with respect to the following:

(i) $a \leq b \Leftrightarrow a \vee b = b$

(ii) $a \vee b$ = least upper bound of a and b where $a, b \in S$.

Definition 1.4. Let S be a semilattice and I, a non-empty subset of S. Then I is called an ideal of S if

(i) $x \in I, y \in I \Rightarrow x \vee y \in I$

(ii) $x \in I, t \in S$ and $t \leq x \Rightarrow t \in I$

Definition 1.5. Let 'a' be an element of a semilattice S. Then the set $\{x \in S/x \leq a\}$ form an ideal of S and it is called principal ideal generated by 'a' and it is denoted by $(a]$.

Theorem 1.2. If $I(S)$ denote set of all ideals of a semilattice S then $I(S)$ is a lattice with respect to the following

(i) $I_1 \leq I_2 \Leftrightarrow I_1 \subseteq I_2$

(ii) $I_1 \vee I_2 = \{x \in S \mid x \leq x_1 \vee x_2$ for some, $x_1 \in I_1, x_2 \in I_2\}$

(iii) $I_1 \wedge I_2 = \{x \in S \mid x \in I_1$, and $x \in I_2\}$

where $I_1, I_2 \in I(S)$.

Theorem 1.3. If S is a semilattice and a, b in S then $(a \vee b] = (a] \vee (b]$

Definition 1.6. Let S be a semilattice and F, a non-empty subset of S. Then F is called dual deal or filter of S if

(i) $x \in F, y \in F \Rightarrow$ there exists $z \in F$ such that $z \leq x, z \leq y$

(ii) $x \in F, t \in S$ and $t \geq x \Rightarrow t \in F$

Then we observe that if $\Im(S)$ denote set of all filters of S then $\Im(S)$ is a lattices with respect to the following:

(1) $F_1 \leq F_2 = F_1 \subseteq F_2$

(2) $F_1 \vee F_2 = \{f \in S/f \geq f_1, f_2$ for some $f_1 \in F_1, f_2 \in F_2\}$

(3) $F_1 \wedge F_2 = \{x \in S/x \in F_1$ and $x \in F_2\}$

where F_1, F_2 in $\Im(S)$.

Definition 1.7. An ideal D of a semilattice S is called distributive ideal if

$$D \vee (X \wedge Y) = (D \vee X) \wedge (D \vee Y), \text{ for all } X, Y \in I(S)$$

Theorem 1.4 (Characterization theorem for distributive ideal)

Let D be an ideal of a semilattice S. Then the following conditions are equivalent

(i) D is distributive

(ii) The map $\phi : X \rightarrow D \vee X$ is a homomorphism of $I(S)$ onto

$$[D) = \{X \text{ in } I(S) \mid X \geq D\}$$

(iii) The binary relation θ_D on $I(S)$ is defined by

$$\text{“}X \equiv Y\ (\theta_D) \Leftrightarrow D \vee X = D \vee Y \text{ where } X, Y \text{ in } I\,(S)\text{”}$$

is a congruence relation.

Definition 1.8. An ideal D of a semilattice S is called standard ideal if

$$X \wedge (D \vee Y) = (X \wedge D) \vee (X \wedge Y), \text{ for all } X, Y, \in I\,(S)$$

Theorem 1.5 (Characterization theorem for standard ideal)

Let D be an ideal of a semilattice S. Then the following conditions are equivalent

(i) D is standard

(ii) The binary relation θ_D on $I\,(S)$ defined by

$$\text{“}X \equiv Y\ (\theta_D) \text{ iff } (X \wedge Y) \vee D_1 = X \vee Y, \text{ for some } D_1 \leq D\text{”}$$

is a congruence relation.

(iii) D is a distributive and for all $X, Y \in I\,(S)$

$$D \wedge X = D \wedge Y,\ D \vee X = D \vee Y \text{ imples } X = Y.$$

Definition 1.9. An ideal D of a semilattice S is called neutral ideal if

$$(D \vee X) \wedge (X \vee Y) \wedge (Y \vee D) = (D \wedge X) \vee (X \wedge Y) \vee (Y \wedge D),$$

for all $X, Y, \in I\,(S)$.

Theorem 1.6. (Characterization theorem for neutral ideals)

Let D be an ideal of a semilattice S. Then the following conditions are equivalent.

(i) D is neutral

(ii) D is distributive, D is dually distributive and $D \wedge X = D \wedge Y$, $D \vee X = D \vee Y$ for all $X, Y \in I\,(S)$ implies $X = Y$.

Distributive subsemilattice

In this section a generalization of distributive ideal for convex subsemilattice called distributive convex subsemilattice or distributive subsemilattices have been introduced and established some examples.

Definition 2.1. Let S be a semilattice and D a non-empty subset of S. Then D is called a convex subsemilattice if

(i) $a, b \in D \Rightarrow a \vee b \in D$

(ii) $x, y \in D, c \in S$ and $x \leq c \leq y \Rightarrow c \in D$.

Definition 2.2. A convex subsemilattice is generated by a subset A of a semilattice S will be denoted by $< A >$

For any two non-empty subsets A and B of a semilattice S, it is defined that

$$A \vee B = < \{a \vee b \mid a \in A, b \in B\} >$$

$$A \wedge B = < \{t \in S \mid t \leq a, t \leq b, a \in A, b \in B\} >$$

That is, $A \vee B$ and $A \wedge B$ are convex subsemilattice of S generated by the element $a \vee b$ and t (where $t \leq a, t \leq b, a \in A, b \in B$) respectively.

Definition 2.3. If X and Y are convex subsemilattices of a semilattice S, then $< X, Y >$ is the smallest convex subsemilattice generated by X and Y and

$$< X, Y > = \{t \in S/u \leq t \leq x \vee y \text{ with } u \in S, u \leq x, y, x \in X, y \in Y\}$$

Definition 2.4. A convex subsemilattice D of a semilattice S is called distributive subsemilattice if

$$< D, X \wedge Y > = < D, X > \wedge < D, Y >$$

$$< D, X \vee Y > = < D, X > \vee < D, Y >$$

hold for any pair of convex subsemilattices X and Y of S whenever $D \cap X \neq \phi$ and $D \cap Y \neq \phi$.

Theorem 2.1. For each $d \in S$, $\{d\}$ is a distributive subsemilattice of S.

Proof: Take $D = \{d\}$

Suppose $\quad D \cap X \neq \phi, D \cap Y \neq \phi$

$$D \cap X \neq \phi \Rightarrow d \in X$$

$$\Rightarrow < D, X > = X \qquad \ldots (1)$$

$$D \cap Y \neq \phi \Rightarrow d \in Y$$

$$\Rightarrow < D, Y > = Y \qquad \ldots (2)$$

$$d \in X, d \in Y \Rightarrow d \in X \wedge Y, \; d \in X \vee Y$$

$$\Rightarrow < D, X \wedge Y > = X \vee Y$$

$$< D, X \wedge Y > = X \wedge Y \qquad \ldots (3)$$

Using (1) and (2) in (3) we get

$$< D, X \wedge Y > = < D, X > \wedge < D, Y >$$

$$< D, X \vee Y > = < D, X > \vee < D, Y >$$

whenever $D \cap X \neq \phi, \; D \cap Y \neq \phi$.

Thus $\{d\}$ is a distributive subsemilattice of S.

Example 2.1. Consider the semilattice $S = \{a_0, a_1, \ldots\ldots, a_n, a, b, c, 1\}$ of the figure 2.1.

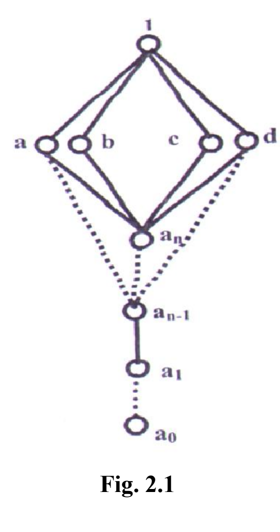

Fig. 2.1

Take $D = \{a_0, a_1 \ldots\ldots\ldots\ldots a_n, a\}$

Then D is a subsemilattice but not distributive.

Consider $X = \{a_0, a_1 \ldots\ldots\ldots a_n, b\}$

$Y = \{a_0, a_1 \ldots\ldots\ldots a_n, c\}$

Then $X \wedge Y = \{a_0, a_1 \ldots\ldots\ldots a_n\}$

$$< D, X \wedge Y > = \{a_0, a_1 \ldots\ldots\ldots a_n, a\} \quad \ldots (1)$$

$$< D, X > = S$$

$$< D, Y > = S$$

$$< D, X > \wedge < D, Y > = S \quad \ldots (2)$$

From (1) and (2) we get

$$< D, X \wedge Y > \neq < D, X > \wedge < D, Y >$$

Thus it is not a distributive subsemilattice.

Characterization theorem

In this section to established some characterization theorems for distributive subsemilattice.

It is evident that

Proposition 3.1. If X, Y are subsets of a semilattice S then $(X \vee Y] = (X] \vee (Y]$, where $(X]$ denotes the ideal generated by X.

Proof: We have $X \leq (X]$, $Y \leq (Y] \Rightarrow X \vee Y \leq (X] \vee (Y]$,

Thus $(X \vee Y] \leq (X] \vee (Y]$ ... (1)

Let $t \in (X] \vee (Y]$ be arbitrary

$\Rightarrow t \leq a \vee b$ for some $a \in (X]$, $b \in (Y]$

$\Rightarrow t \leq a \vee b$ for some $a \leq x, x \in X, b \leq y, y \in Y$

$\Rightarrow t \leq a \vee b, a \vee b \in (X \vee Y]$

$\Rightarrow t \in (X \vee Y]$

Therefore $(X] \vee (Y] \leq (X \vee Y]$ … (2)

From (1) and (2) we have

$$(X \vee Y] = (X] \vee (Y]$$

Proposition 3.2. If A and B are convex subsemilattice of a semilattice S, then

(i) $(A \wedge B] = (A] \wedge (B]$

(ii) $< A, (B] > = (A] \vee (B]$

Proof : For (i)

We have $A \leq (A]$, $B \leq (B]$

$\Rightarrow A \wedge B \leq (A] \wedge (B]$

$\Rightarrow (A \wedge B] \leq (A] \wedge (B]$ … (1)

Let $x \in (A] \wedge (B]$ be arbitrary

$\Rightarrow x \in (A]$ and $x \in (B]$

$\Rightarrow x \leq a_1$, for some $a_1 \in A$ and

$x \leq b_1$, for some $b_1 \in B$

$\Rightarrow x \leq a_1, b_1$ with $a_1 \in A, b_1 \in A$

$\Rightarrow x \in (A \wedge B]$

Thus $\quad (A] \wedge (B] \leq (A \wedge B]$ … (2)

From (1) and (2) we have

$$(A] \wedge (B] = (A \wedge B]$$

For (ii)

We have $A \leq (A]$ and $B \leq (B]$

$\Rightarrow < A, (B] > \; \leq \; < (A], (B] > = (A] \vee (B]$

Therefore $\quad < A, (B] > \; \leq \; (A] \vee (B]$ … (3)

Let $x \in (A] \vee (B]$ be arbitrary.

$\Rightarrow x \leq a \vee b$, for some $a \in (A], b \in (B]$

$\Rightarrow x \leq a \vee b$, for some $a \leq a_1, a_1 \in A, b \in (B]$

$\Rightarrow x \leq a_1 \vee b$, for some $a_1 \in A, b \in (B]$

$\Rightarrow t \leq x \leq a_1 \vee b$, with $t \leq a_1, t \leq b$

$\Rightarrow x \in < A, (B] >$

Therefore $\quad (A] \vee (B] \; \leq \; < A, (B] >$ … (4)

From (3) and (4) we have

$$< A, (B] > = (A] \vee (B]$$

Theorem 3.1. An ideal D of a semilattice S is distributive iff it is a distributive subsemilattice of S.

Proof: Assume that an ideal D is a distributive ideal of a semilattice S.

To prove D is a distributive subsemilattice.

Let X and Y be two arbitrary convex subsemilattice of S.

Then $\quad < D, X \wedge Y) = D \vee (X \wedge Y]$, since $< X, (Y] > = (X] \vee (Y]$

$= D \vee [(X] \wedge (Y]]$, since $(X \wedge Y] = (X] \wedge (Y]$

$= (D \vee (X]) \wedge (D \vee (Y])$, by our assumption

$= < D, X > \wedge < D, Y >$

Claim: $\quad < D, X \vee Y > = < D, X > \vee < D, Y >$

We have

$$D \leq < D, X >, \quad D \leq < D, Y >$$

$\Rightarrow D \vee D \leq < D, X > \vee < D, Y >$

$\Rightarrow D \leq < D, X > \vee < D, Y >$ … (1)

Also $X \leq\ < D, X >, Y \leq\ < D, Y >$

$$\Rightarrow X \vee Y \leq\ < D, X > \vee < D, Y > \quad \ldots (2)$$

From (1) and (2) we have

$$< D, X \vee Y > \leq\ < D, X > \vee < D, Y > \quad \ldots (3)$$

Clearly $< D, X > \vee < D, Y >$ is a convex subsemilattice generated by the elements of the form $(d_1 \vee x_1) \vee (d_2 \vee y_1)$ where $d_1, d_2 \in D, x_1 \in (X], y_1 \in (Y]$

Now,

$$x_1 \in (X], \quad y_1 \in (Y]$$

$$\Rightarrow x_1 \leq x, y_1 \leq y \text{ with } x \in X, y \in Y$$

$$\Rightarrow x_1 \vee y_1 \leq x \vee y, \ x \vee y \in X \vee Y$$

$$\Rightarrow x_1 \vee y_1 \in (X \vee Y]$$

$$\Rightarrow x_1 \vee y_1 \in\ < D, X \vee Y >, \text{ since } < D, X \vee Y > = D \vee (X \vee Y]$$

Also $d_1, d_2 \in D \Rightarrow d_1 \vee d_2 \in D$

$$\Rightarrow d_1 \vee d_2 \in\ < D, X \vee Y >,$$

Therefore $(d_1 \vee d_2) \vee (x \vee y) \in\ < D, X \vee Y >$

Now,

$$x_1 \vee y_1 \leq (d_1 \vee x_1) \vee (d_2 \vee y_1), \text{ since } < D, X > = D \vee (X]$$

$$\text{and } < D, Y > = D \vee (Y]$$

$$\leq (d_1 \vee x) \vee (d_2 \vee y), \text{ since } x_1 \leq x, y_1 \leq y, \ x \in X, y \in Y$$

$$= d_1 \vee (x \vee (d_2 \vee y))$$

$$= d_1 \vee ((x \vee d_2) \vee y)$$

$$= d_1 \vee ((d_2 \vee x) \vee y)$$

$$= (d_1 \vee d_2) \vee (x \vee y)$$

Hence, $x_1 \vee y_1 \leq (d_1 \vee x_1) \vee (d_2 \vee y_1) \leq (d_1 \vee d_2) \vee (x \vee y)$

$$\text{with } x_1 \vee y_1, \ (d_1 \vee d_2) \vee (x \vee y) \in\ < D, X \vee Y >$$

$$\Rightarrow (d_1 \vee x_1) \vee (d_2 \vee y_1) \in\ < D, X \vee Y > \text{ by convexity}$$

Therefore $< D, X > \vee < D, Y > \leq\ < D, X \vee Y >$ $\quad \ldots (4)$

From (3) and (4) we have

$$< D, X \vee Y > = < D, X > \vee < D, Y >$$

Thus $< D, X \wedge Y > = < D, X > \wedge < D, Y >$

$$< D, X \vee Y > = < D, X > \vee < D, Y >$$

whenever $D \cap X \neq \phi, \ D \cap Y \neq \phi$.

Hence D is a distributive subsemilattice of S.

Conversely, assume that an ideal D is a distributive subsemilattice of S.

To prove D is a distributive ideal.

Let X and Y be two arbitrary ideals of S.

Then X and Y are convex subsemilattice of S.

Moreover $D \cap X \supseteq \{0\}$ and $D \cap Y \supseteq \{0\}$

$$\Rightarrow D \cap X \neq \phi \text{ and } D \cap Y \neq \phi$$

They by our assumption

$$< D, X \wedge Y > = < D, X > \wedge < D, Y > \quad \dots (5)$$

$$< D, X \vee Y > = < D, X > \vee < D, Y > \quad \dots (6)$$

whenever $D \cap X \neq \phi, D \cap Y \neq \phi$

We have $< X, Y > = X \vee Y \quad \dots (7)$

Using (7) in (5) we get

$$D \vee (X \wedge Y) = (D \vee X) \wedge (D \vee Y) \text{ for all } X, Y \in I(S)$$

$\Rightarrow D$ is distributive ideal.

Corollary 3.1. Every standard ideal in a semilattice is a distributive subsemilattice.

Proof : Follows from theorem 1.5 and 3.1.

Corollary 3.2. Every neutral ideal in a semilattice is distributive subsemilattice.

Proof : Follows from theorem 1.6 and 3.1.

Theorem 3.2. A dual ideal D' of a semilattice S is distributive if any only if it is distributive subsemilattice.

Proof : Follows dually.

Theorem 3.3. If D is a distributive subsemilattice of a semilattice S then the binary relation θ_D defined by

"$x \equiv y\ (\theta_D)$ iff $x \vee t \equiv y \vee t$, for some $t \in D$"

is a congruence relation on S.

Proof : (i) θ_D is reflexive :

Let $x \in S$ be arbitrary

Then $x \vee t = x \vee t$, for some $t \in D$

$\Rightarrow x \equiv x\ (\theta_D)$, for all $x \in S$

(ii) θ_D is symmetric : Let $x, y \in S$ be arbitrary

Suppose $x \equiv y\ (\theta_D)$

$\Rightarrow x \vee t = y \vee t$, for some $t \in D$

$\Rightarrow y \vee t = x \vee t$, for some $t \in D$

$\Rightarrow y \equiv x\ (\theta_D)$, for all $x, y \in S$

Thus $x \equiv y\ (\theta_D) \Rightarrow y \equiv x\ (\theta_D)$, for all $x, y \in S$.

(iii) θ_D is transitive : Let $x, y, z \in S$ be arbitrary

Suppose $x \equiv y\ (\theta_D)$ and $y \equiv z\ (\theta_D)$

$\Rightarrow x \vee t_1 = y \vee t_1$, for some $t_1 \in D$ and

$y \vee t_2 = z \vee t_2$, for some $t_2 \in D$

$\Rightarrow \quad x \vee (t_1 \vee t_2) = (x \vee t_1) \vee t_2$

$= z \vee (t_1 \vee t_2)$ for some $t_1 \vee t_2 \in D$

$\Rightarrow \quad x \equiv z\ (\theta_D)$, for all $x, y, z \in S$.

Thus $x \equiv y\ (\theta_D)$ and $y \equiv z\ (\theta_D) \Rightarrow x \equiv z\ (\theta_D)$, for all $x, y, z \in S$.

(iv) Substitution Property : Let $x, x_1, y, y_1 \in S$ be arbitrary.

Suppose $x \equiv x_1(\theta_D)$ and $y \equiv y_1\ (\theta_D)$

$\Rightarrow x \vee t_1 = x_1 \vee t_1$, for some $t_1 \in D$ and

$Y \vee t_2 = y_1 \vee t_2$, for some $t_2 \in D$

$\Rightarrow (x \vee y) \vee (t_1 \vee t_2) = x \vee [y \vee (t_1 \vee t_2)]$

$= (x_1 \vee y_1) \vee (t_1 \vee t_2)$, for some $t_1 \vee t_2 \in D$

$\Rightarrow x \vee y \equiv x_1 \vee y_1\ (\theta_D)$, for all $x, y, x_1, y_1 \in S$

Thus $x \equiv x_1\ (\theta_D)$ and $y \equiv y_1\ (\theta_D)$

$\Rightarrow x \vee y \equiv x_1 \vee y_1\ (\theta_D)$

Hence θ_D is a congruence relation.

Theorem 3.4. If D is a convex subsemilattice of S such that the relation θ_D is defined by "$x \equiv y\ (\theta_D)$ iff $x \vee t = y \vee t$ for some $t \in D$" is a congruence relation then D is a distributive subsemilattice S.

Proof: Let D be a convex subsemilattice of S with the relation θ_D is defined by "$x \equiv y\ (\theta_D)$ iff $x \vee t = y \vee t$ for some $t \in D$" is a congruence relation.

To prove that D is a distributive subsemilattice of S.

It is sufficient to verify that

(i) $\quad < D, X \wedge Y > = < D, X > \wedge < D, Y >$

(ii) $\quad < D, X \vee Y > = < D, X > \vee < D, Y >$

for any two convex subsemillattice X, Y of S whenever $D \cap X \neq \phi$, $D \cap Y \neq \phi$.

For (1) : We have

$D \leq < D, X >, \quad D \leq < D, Y >$

$\Rightarrow D \leq < D, X > \vee < D, Y >$

Also $\quad X \leq < D, X >, \; Y \leq < D, Y >$

$\Rightarrow X \vee Y \leq < D, X > \vee < D, Y >$

Therefore $\quad < D, X \vee Y > \leq < D, X > \vee < D, Y > \quad \ldots (1)$

Let $p \in < D, X > \vee < D, Y >$ be arbitrary.

$\Rightarrow \quad p = a \vee b$ where $a \in < D, X >, b \in < D, Y >$

$\Rightarrow \quad p = a \vee b$ where $q \leq a \leq d_1 \vee x_1$ where $q \leq d_1$, $d_1 \in D$, $q \leq x_1$,

$x_1 \in X$, $r \leq b \leq d_2 \vee y_1$ where $r \leq d_2$, $d_2 \in D$, $r \leq y_1$, $y_1 \in Y$.

$\Rightarrow \quad p = a \vee b$ where $q \vee r \leq a \vee b \leq (d_1 \vee d_2) \vee (x_1 \vee y_1)$

with $q \vee r \leq d_1 \vee d_2$, $(d_1 \vee d_2) \in D$ $q \vee r \leq x_1 \vee y_1$, $x_1 \vee y_1 \in X \vee Y$

$\Rightarrow \quad p = a \vee b, a \vee b \in \langle D, X \vee Y \rangle$

$\Rightarrow \quad p \in \langle D, X \vee Y \rangle$

Thus $\quad \langle D, X \rangle \vee \langle D, Y \rangle \leq \langle D, X \vee Y \rangle \quad \ldots (2)$

From (1) and (2) we have

$$\langle D, X \vee Y \rangle = \langle D, X \rangle \vee \langle D, Y \rangle$$

For (2) : We have $\quad D \leq \langle D, X \rangle,\ D \leq \langle D, Y \rangle$

$\Rightarrow D \leq \langle D, X \rangle \wedge \langle D, Y \rangle$

Also, $\quad X \leq \langle D, X \rangle,\ Y \leq \langle D, Y \rangle$

$\Rightarrow X \wedge Y \leq \langle D, X \rangle \wedge \langle D, Y \rangle$

Therefore $\quad \langle D, X \wedge Y \rangle \leq \langle D, X \rangle \wedge \langle D, Y \rangle \quad \ldots (3)$

Let $p \in \langle D, X \rangle \wedge \langle D, Y \rangle$ be arbitrary

$\Rightarrow p \in \langle D, X \rangle$ and $p \in \langle D, Y \rangle$

$\Rightarrow q \leq p \leq d_1 \vee x_1$ where $q \leq d_1, d_1 \in D,\ q \leq x_1, x_1 \in X$

and $r \leq p \leq d_2 \vee y_1$, where $r \leq d_2, d_2 \in D, r \leq y_1, y_1 \in Y$

$\Rightarrow (q \vee r) \leq p \leq (d_1 \vee x_1) \vee (d_2 \vee y_1) = (d_1 \vee d_2) \vee x$

with $x = x_1 = y_1,\ x \in X \wedge Y$

$\Rightarrow (q \vee r) \leq p \leq d \vee x$ with $d = d_1 \vee d_2,\ d_1 \vee d_2 \in D,\ x \in X \wedge Y$

$\Rightarrow (q \vee r) \leq p \leq d \vee x$, where $q \vee r \leq d, d \in D, q \vee r \leq x,\ x \in X \wedge Y$

$\Rightarrow p \in \langle D, X \wedge Y \rangle$

Therefore $\quad \langle D, X \rangle \wedge \langle D, Y \rangle \leq \langle D, X \wedge Y \rangle \quad \ldots (4)$

From (3) and (4) we have

$$\langle D, X \wedge Y \rangle = \langle D, X \rangle \wedge \langle D, Y \rangle$$

Thus D is a distributive convex subsemilattice.

From theorem 3.3 and 3.4 we have

Theorem 3.5. A convex subsemilattice D is a distributive subsemilattice of S

$\Leftrightarrow \theta_D$ is a congruence relation defined above.

Corollary 3.3. If D is a distributive subsemilattice of a semilattice S then D is a congruence class by a congruence relation θ_D.

Proof: Let $\quad x \equiv y\ (\theta_D),\ x \leq y$

We have to prove that if one of these elements belongs to D then both are in D.

Assume that $\quad x \in D$

To prove $\quad y \in D$

By the definition of θ_D, $x \vee t = y \vee t$, for some $t \in D$

$$x, t \in D \Rightarrow x \vee t \in D$$

We have, $x \leq y \leq y \vee t \leq x \vee t$ with $x \vee t \in D,\ x \in D$

$\Rightarrow y \in D$, by the convexity of D

Hence $x \equiv y\ (\theta_D)$, $x \leq y$ and $x \in D \Rightarrow y \in D$

Thus D is a congruence class by a congruence relation θ_D,

Corollary 3.4. If D_1 and D_2 are two distributive subsemilattices then $D_1 \cap D_2$ is either a distributive subsemilattice or it is empty.

Proof : Let D_1 and D_2 are two distributive subsemilattices.

To prove that $D_1 \cap D_2$ is either a distributive subsemilattice or it is empty.

Suppose $\quad D_1 \cap D_2 \neq \phi$

Assume $\quad u \in D_1 \cap D_2$

Let $\quad x \equiv y\ (\theta_{D1} \cap \theta_{D2})$

$\Rightarrow \quad x \equiv y\ (\theta_{D1})$ and $x \equiv y\ (\theta_{D2})$

$\Rightarrow \quad x \vee t_1 = y \vee t_1$ for some $t_1 \in D_1$ and $x \vee t_2 = y \vee t_2$ for some $t_2 \in D_2$

with $t_1 \leq u$, $t_2 \leq u$

$\Rightarrow \quad x \vee (t_1 \vee t_2) = y \vee (t_1 \vee t_2)$ for some $t_1 \vee t_2 \leq u$

$\Rightarrow \quad x \vee (t_1 \vee t_2) = y \vee (t_1 \vee t_2)$ with $t_1 \vee t_2 \in D_1 \cap D_2$

$\Rightarrow \quad x \equiv y\ (\theta_{D1 \cap D2})$

$\Rightarrow \quad D_1 \cap D_2$ is a distributive subsemilattice.

Corollary 3.5. The meet of a distributive ideal and distributive dual ideal is a convex subsemilattice (provided it is non-empty)

Proof: We know that all distributive ideals and distributive dual ideals are distributive subsemilattices. Hence by the above corollary we get this result.

Proposition 3.1. Let D be a distributive subsemilattice and I be an arbitrary convex subsemilattice such that $I \cap D \neq \phi$. Then $I \cap D$ is not a distributive convex subsemilattice.

Proof: By an example, consider the following lattice figure

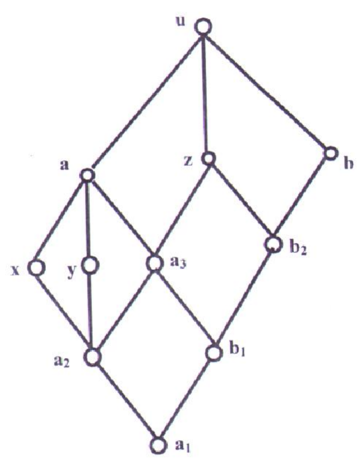

Fig. 3.1

Take $\quad D = \{a_1, b_1, b_2, b\}$

$I = \{a_1, a_2, a_3, b_1, x, y, a\}$

Then $I \cap D = \{a_1, b_1\} \neq \phi$ is not a distributive convex subsemilattice.

Let $$X = \{a_1, a_2, x\}$$
$$Y = \{a_1, a_2, y\}$$

Then $$X \wedge Y = \{a_1, a_2\}$$

$$< I \cap D, X \wedge Y > \; = \{a_1, a_2, b_1\} \quad \ldots (1)$$

$$< I \cap D, X > \; = \{a_1, a_2, a_3, x, y, b_1\}$$

$$< I \cap D, Y > \; = \{a_1, a_2, a_3, x, y, b_1\}$$

$$< I \cap D, X > \wedge < I \cap D, Y > = \{a_1, a_2, a_3, x, y, b_1\} \quad \ldots (2)$$

From (1) and (2) we get

$$< I \cap D, X \wedge Y > \; \neq \; < I \cap D, X > \; \wedge < I \cap D, Y >$$

Hence $I \cap D$ is not a distributive convex subsemilattice.

Conclusion

In the case of distributive ideal Gratzer's problem is solved.

References

1. Chellappa, B. and Natarajan, R., Dually Distributive convex sublattice, *Acta Ciencia Indica*, Vol **XXXIII** M, No. **3**, 1127-1141 (2007).
2. Chellappa, B. and Natarajan, R., Dually standard convex sublattice, *Acta Ciencia Indica*, Vol **XXXIII** M, No. **2**, 527-538 (2007).
3. Gratzer, G., *General Lattice Theory*, Academic Press Inc. (1978).
4. Natarajan, R., Distributive Convex sublattice, *Acta Ciencia Indica*, Vol. **XXX** M, No. **3**, 527-532 (2004).
5. Natarajan, R. and Vairavan. L., Distributive ideals in semilattice, *Acta Ciencia Indica*, Vol. **XXXI** M, No. **2**, 401-410 (2005).
6. Natarajan, R. and Vairavan. L., Standard ideals in semilattice, *Acta Ciencia Indica*, Vol. **XXXI**, No. **4**, 1041-1052 (2005).

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