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ON EXISTENCE RESULT OF IMPULSIVE ANTIPERIODIC BOUNDARY VALUE PROBLEM OF FRACTIONAL ORDER $q \in (1, 2)$

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This paper is motivated from some recent papers on impulsive fractional differential equation. In this paper an antiperiodic boundary value problem for an impulsive fractional differential equation of order $q \in (1, 2)$ is studied. We develop an effective way to find solution of such type of problems. A special hybrid singular type Grownwell inequality is established to obtain prior bounds of the solution. The sufficient conditions for existence of the solutions are established by applying fixed point methods under the mixed nonlinear *D*-contraction condition, comparison condition, sublinear growth condition for nonlinear term. Examples are given to illustrate the results.

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INTRODUCTION

In recent years the subject of fractional calculus gained much momentum and attracted many researchers and mathematicians. Considerable interest in field of fractional calculus has been developed by the applications to different areas of applied science and engineering like physics, biophysics, aerodynamics, control theory, viscoelasticity, capacitor theory, electrical circuit, description of memory and hereditary properties etc. Remarkable monographs are available which provide the main theoretical tools for the qualitative analysis of fractional differential equations, see [l]-[7]. Meanwhile many evolution processes are subject to short term perturbations whose duration is negligible in comparison with the duration of processes, that is in form of impulses. A strong motivation for studying impulsive fractional differential equations comes from the fact they have been proved to be valuable tool in a number of fields such as physics, geophysics, regular variation in thermodynamics, electrical circuits etc. For more details one can see the monographs and research papers and references therein, see [8]-[21].

In order to describe mass-spring-damper system subject to abrupt changes as well as other phenomena such as earthquake, it is natural to use impulsive fractional differential systems to describe such problems. Li, Chen, Li [20] studied generalized antiperiodic boundary value problem of impulsive fractional differential equations. Motivated by their work and mass-spring-damper system in [22, 23], we will study the following antiperiodic boundary value problems for impulsive fractional differential equation.

$$\left.\begin{aligned} &{}^{c}D_t^{q}u(t) = f(t, u(t)),\ t \in J' = J \setminus \{t_1, t_2, \ldots t_m\},\ J = [0, 1], \\ &\Delta u(t_k) = I_k,\ \Delta u'(t_k) = J_K,\ k = 1, 2, \ldots m. \\ &3u(0) = -u(1),\ 3u'(0) = -u'(1). \end{aligned}\right\} \qquad \ldots (1.1)$$

where ${}^{c}D_t^{q}$ denotes the Caputo's fractional derivative of order $q \in (1, 2)$ with lower limit zero. $f : J \times R \to R$ is jointly continuous, $I_k, J_k \in R$ and t_k satisfy $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = 1$, $\Delta u(t_k) = u(t_k^{+}) - u(t_k^{-})$ with $u(t_k^{+}) = \lim_{\varepsilon \to 0^{+}} u(t_k + \varepsilon)$ and $u(t_k^{-}) = \lim_{\varepsilon \to 0^{-}} u(t_k + \varepsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$.

The first purpose of this paper is to find a natural formula of solution for the problem (1.1). A better definition of the solution for impulsive fractional differential equation is introduced. The second purpose is to establish a sufficient condition for existence of solution under the mixed nonlinear 'D-contraction condition, comparison condition, sublinear growth condition and nonlinear growth condition via different fixed point methods. Meanwhile we emphasize that a new special hybrid singular type Gronwell inequality is given to obtain a prior bounds of the solutions.

MATHEMATICAL PRELIMINARIES

In this section we introduce notations, definitions and preliminary facts. Throughout this paper, let $C(J, R)$ be the Banach space of all continuous functions from J into R with the norm $\| u \|_c = \sup\{| u(t) | : t \in J\}$ for $u \in C(J, R)$. We also define $PC(J, R) = \{u : J \to R : u \in (t_k, t_{k+1}), R\}$, $k = 0, 1, \ldots m$ and there exist $u(t_k^{+})$ and $u(t_k^{-})$, $k = 1, 2, \ldots m$ with $u(t_k^{-}) = u(t_k)$ with the norm $\| u \|_{pc} = \sup\{| u(t) | : t \in J\}$. For measurable functions $I : J \to R$, define the norm $\| l \|_{L^{\sigma}(J, R)} = (\int_J |l(t)|^{\sigma} dt)^{1/\sigma}$, $1 \le \sigma < \infty$. We denote $L^{\sigma}(J, R)$ the Banach space of all Lebesgue measurable functions l with $\| l \|_{L^{\sigma}} < \infty$.

Let us recall some more definitions of fractional calculus. For more details see [2].

Definition 2.1. The fractional integral of order 7 with the lower limit zero for a function $f : [0, \infty) \to R$ is defined as

$$I_t^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds,\ t > 0,$$

provided that right is point-wise defined on $[0, \infty)$, where $\Gamma(.)$ is the gamma function.

Definition 2.2. (See [2]). The Riemann-Liouville derivative of order γ with the lower limit zero for a function $f : [0, \infty) \to R$ can be written as

$$ {}^{L}D_t^{\gamma} f(t) = \frac{1`}{\Gamma(n-\gamma)\, dt^n} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \ t > 0, \ n-1 < \gamma < n. $$

Next we introduce the Caputo fractional derivative.

Definition 2.3. (See [2]). Caputo fractional derivative of order γ for a function $f : [0, \infty) \to R$ is defined as

$$ {}^{C}D_t^{\gamma} f(t) = {}^{L}D_t^{\gamma} f(t) \left[f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \right], \ t > 0, \ n-1 < \gamma < n. $$

Lemma 2.4. *(See [27]). For $q > 0$ the general solution of fractional differential equation ${}^{c}D_t^{q} u(t) = 0$ is given by*

$$ u(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}. $$

where $c_i \in R$, $i = 0, 1, 2, \ldots, n-1$, $(n = -[-q])$ and $[q]$ denotes the integer part of real number $q > 0$.

Remark 2.5. In view of Lemma 2.4, it follows that

$$ I^{q} ({}^{c}D_t^{q} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1} $$

where $c_i \in R$, $i = 0, 1, 2, \ldots, n-1$, $(n = -[-q])$

Definition 2.6. (See [29]). Let X be a infinite dimensional Banach space with the norm $\|.\|$. A mapping $T : X \to X$ is called D-Lipshitzian if there exists a continuous non-decreasing function $\phi_T : R^+ \to R^+$ satisfying

$$ \| T_x - T_y \| \le \phi_T (\| x - y \|) $$

for all $x, y \in X$ with $\phi_T(0) = 0$. Sometime we call the function ϕ_T a D-function of T on X. If $\phi_T(r) = \alpha r$ for some constant $\alpha > 0$ then T is called a Lipschitzian with a Lipschitz constant α and further if $\alpha < 1$, then T is called a contraction with the contraction constant α. Again if ϕ_T satisfies $\phi_T(r) < r, \ r > 0$ then T is called a nonlinear D-contraction on X.

Remark 2.7. It is clear that every D-contraction implies nonlinear contraction and nonlinear contraction implies D-Lipschitzian but the reverse implication may not hold.

Now following generalized Gronwell inequality which is introduced by Wang *et al.* [25] can be used in the fractional differential equation with initial condition.

Lemma 2.8. *(Lemma 2, [25]). Let $y \in C(J, R)$ is satisfying following inequality*

$$ y(t) \le c_{1+c_2} \int_0^t | y(s) |^{\lambda_1} ds + c_3 \int_0^1 | y(s) |^{\lambda_2} ds, \ t \in J $$

where $\lambda_1 \in [0, 1]$, $\lambda_2 \in [0, 1]$, $c_1, c_2, c_3 \ge 0$ are contants. Then there exists a constant $M^* > 0$ such that

$$ | y(t) | \le M^* | $$

Using Lemma 2.8, we can obtain a new special hybrid singular type Gronwell inequality.

Lemma 2.9. *Let* $y \in C(J, R)$ *is satisfying following inequality*

$$y(t) \le c_1 + c_2 \int_0^t (t-s)^{q-1} |y(s)|^{\lambda} ds + c_3 \int_0^1 (1-s)^{q-1} |y(s)|^{\lambda} ds$$

$$+ c_4 \int_0^1 (1-s)^{q-2} |y(s)|^{\lambda} ds \quad \dots (2.1)$$

where $q \in (1, 2)$, $\lambda \in \left[0, 1 - \frac{1}{p}\right)$ for some $1 < p < \frac{1}{2-q}$, $c_1, c_2, c_3, c_4 \ge 0$ are contants. Then there exists a constant $M^* > 0$ such that

$$|y(t)| \le M^*$$

Proof. Let

$$x(t) = \begin{cases} 1, & |y(t)| \le 1, \\ y(t) & |y(t)| > 1. \end{cases}$$

It follows from condition (2.1) and Holder's inequality that

$$|y(t)| \le |x(t)| \le (c_1 + 1) + c_2 \int_0^t (t-s)^{q-1} |x(s)|^{\lambda} ds + c_3 \int_0^1 (1-s)^{q-1} |x(s)|^{\lambda} ds$$

$$+ c_4 \int_0^1 (1-s)^{q-2} |x(s)|^{\lambda} ds$$

$$\le (c_1 + 1) + c_2 \left(\int_0^t (t-s)^{p(q-1)} ds \right)^{\frac{1}{v}} \left(\int_0^t |x(s)|^{\frac{\lambda p}{p-1}} ds \right)^{\frac{p-1}{v}}$$

$$+ c_3 \left(\int_0^1 (1-s)^{p(q-1)} ds \right)^{\frac{1}{p}} \left(\int_0^t |x(s)|^{\frac{\lambda p}{p-1}} ds \right)^{\frac{p-1}{v}}$$

$$+ c_4 \left(\int_0^t (1-s)^{p(q-2)} ds \right)^{\frac{1}{p}} \left(\int_0^t |x(s)|^{\frac{\lambda p}{p-1}} ds \right)^{\frac{p-1}{v}}$$

$$\le (c_1 - 1) + c_2 \left(\frac{1}{p(q-1)+1} \right)^{\frac{1}{p}} \int_0^t |x(s)|^{\frac{\lambda p}{p-1}} ds + (c_3 + c_4) \left(\frac{1}{p(q-2)+1} \right)^{\frac{1}{p}} \int_0^t |x(s)|^{\frac{\lambda p}{p-1}} ds.$$

By lemma 2.8 one can complete the rest proof immediately.

Now we collect *PC*-type Arzela-Ascoli theorem.

Theorem 2.10. *See (Theorem 2.1 [26]). Let X be a Banach space and* $W \subset PC(J, X)$*. If the following conditions are satisfied*

(1) W is uniformly bounded subset of PC (J, X);

(2) W is equicontinuous in (t_k, t_{k+1}), $k = 0, 1, 2 \dots, m$, *where* $t_0 = 0$ *and* $t_{m+1} = 1$;

(3) $W(t) = \{u(t) : u \in W, t \in J \setminus \{t_1, t_2, \dots, t_n\}\}, W(t_k^+) = \{u(t_k^+) : u \in W\}$ *and*

$W(t_k^-) = \{u(t_k^-) : u \in W\}$ *are relatively compact sets of X.*

Then W is a relatively compact subset of PC (J, X).

NATURAL SOLUTION

In this section we derive a natural formula of the solution to the impulsive fractional differential equation of order $q \in (1.2)$.

Lemma 3.1. *A function u is given by*

$$u(t) = \begin{cases} \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1} h(s)\,ds - \frac{1}{4\Gamma q}\int_0^1 (1-s)^{q-1} h(s)\,ds - \frac{(4t-1)}{16\Gamma(q-1)}\int_0^1 (1-s)^{q-2} h(s)\,ds \\ -\frac{1}{16}\sum_{i=1}^m J_i[3+4(t-t_i)] - \frac{1}{4}\sum_{i=1}^m I_i, \text{ for } t \in [0, t_1) \\ \vdots \\ \frac{1}{\Gamma q}\int_0^t (t-s)^{q-1} h(s)\,ds - \frac{1}{4\Gamma q}\int_0^1 (1-s)^{q-1} h(s)\,ds - \frac{(4t-1)}{16\Gamma(q-1)}\int_0^1 (1-s)^{q-2} h(s)\,ds \\ -\frac{1}{16}\sum_{i=1}^m J_i[3+4(t-t_i)] - \frac{1}{4}\sum_{i=1}^m I_i + \sum_{i=1}^k I_i + \sum_{i=1}^k J_i(t-t_i) \\ \text{for } t \in (t_k, t_{k+1}],\ k = 1, 2, \ldots, m \end{cases} \quad \ldots (3.1)$$

is the unique solution of following impulsive problem

$$\left.\begin{aligned} &{}^c D_t^q u(t) = h(t),\ t \in J',\ q \in (1, 2), \\ &\Delta u(t_k) = I_k,\quad \Delta u'(t_k) = J_K,\ k = 1, 2, \ldots, m. \\ &3u(0) = -u(1),\ 3u'(0) = -u'(1),\ a \ge b > 0. \end{aligned}\right\} \quad \ldots (3.2)$$

where $h : J \to R$ is continuous.

Proof. A general solution u of the 1st equation of (3.2) on each interval $(t_{k,}\ t_{k+1})$ (k = 0, 1, ..., m) is given by

$$u(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} h(s)\,ds - c_k - d_k t,\ \text{ for } t \in (t_k, t_{k+1}) \quad \ldots (3.3)$$

where $t_0 = 0$ and $t_{m+1} = 1$, then we have

$$u'(t) = \frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2} h(s)\,ds - d_k, \text{ for } t \in (t_k, t_{k+1}).$$

Applying the condition (3.2) we can find that

$$\frac{1}{\Gamma q}\int_0^1 (1-s)^{q-1} h(s)\,ds - 3c_0 - c_m - d_m = 0 \quad \ldots (3.4)$$

$$\frac{1}{\Gamma(q-1)}\int_0^1 (1-s)^{q-2} h(s)\,ds - 3_0 - d_m = 0 \quad \ldots (3.5)$$

Next, using the right impulsive condition of (3.2), we derive

$$c_{k-1} - c_k = I_k - J_k t_k \quad \dots (3.6)$$

$$d_{k-1} - d_k = J_k \quad \dots (3.7)$$

Applying (3.5) and (3.7), we obtain

$$d_0 = \frac{1}{4\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)\, ds + \frac{1}{4} \sum_{i=1}^{m} J_i \quad \dots (3.8)$$

$$d_m = \frac{1}{4\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)\, ds - \frac{1}{4} \sum_{i=1}^{m} J_i \quad \dots (3.9)$$

So by (3.7) we obtain

$$d_k = d_0 - \sum_{i=1}^{k} J_i \quad \dots (3.10)$$

$$= \frac{1}{4\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)\, ds + \frac{1}{4} \sum_{i=1}^{m} J_i - \sum_{i=1}^{k} J_i$$

Applying (3.4), (3.6) and (3.9), we derive

$$c_0 = \frac{1}{4\Gamma q} \int_0^1 (1-s)^{q-1} h(s)\, ds - \frac{1}{16\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)\, ds$$

$$+ \frac{3}{16} \sum_{i=1}^{m} J_i + \frac{1}{4} \sum_{i=1}^{m} (I_i - J_i t_i) \quad \dots (3.11)$$

So by (3.6) we obtain

$$c_k = c_0 - \sum_{i=1}^{k} (I_i - J_i t_i)$$

$$= \frac{1}{4\Gamma q} \int_0^1 (1-s)^{q-1} h(s)\, ds - \frac{1}{16\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)\, ds + \frac{3}{16} \sum_{i=1}^{m} J_i - \frac{1}{4} \sum_{i=1}^{m} I_i - \sum_{i=1}^{k} I_i$$

$$- \frac{1}{4} \sum_{i=1}^{m} J_i t_i + \sum_{i=1}^{k} J_i t_i \quad \dots (3.12)$$

Hence for $k = 1, 2 \dots m$, (3.10) and (3.12) imply

$$c_k + d_k t = \frac{1}{4\Gamma q} \int_0^1 (1-s)^{q-1} h(s)\, ds + \frac{(4t-1)}{16\Gamma(q-1)} \int_0^1 (1-s)^{q-2} h(s)\, ds + \frac{1}{16} \sum_{i=1}^{m} J_i [3 + 4(t - t_i)]$$

$$+ \frac{1}{4} \sum_{i=1}^{m} I_i - \sum_{i=1}^{k} I_i - \sum_{i=1}^{m} J_i (t - t_i). \quad \dots (3.13)$$

Now it is clear that (3.8), (3.11) and (3.13) imply (3.1).

MAIN RESULTS

This section deals with existence and uniqueness of solutions for the problem (1.1).

Definition 4.1. A function $u \in PC(J, R)$ is said to be a solution of the problem (1.1) if $u(t) = u_k(t)$ for $t \in (t_k, t_{k+1})$ and $u_k \in C([0, t_{k+1}], R)$ satisfies ${}^c D_t^q u_k(t) = f(t, u_k(t))$ a.e. on $(0, t_{k+1})$ with the restriction of $u_{k+1}(t)$ on $[0, t_{k+1}]$ is just $u_k(t)$ and the condition $\Delta u(t_k) = J_k$, $\Delta u'(t_k) = I_k$, $k = 1, 2, \ldots m$ with $3u(0) = -u(1)$ and $3u'(0) = -u'(1)$.

Before stating and proving the main result we introduce following hypotheses

(H1) : $f : f \times R \to R$ is jointly continuous.

(H2) : f satisfies nonlinear D-contraction on the second variable *i.e.* there exists a continuous nondecreasing function $\phi : R^+ \to R^+$ such that

$$|f(t, u) - f(t, v)| \le \phi(|u - v|) \text{ with } \phi(0) = 0 \text{ and } \phi(r) < r.$$

Now we are ready to state our first result in this paper.

Theorem 4.2. *Assume that (H1) with* $\phi(0) = 0$ *and (H2) hold if*

$$\frac{20 + 3q}{16\Gamma(q+1)} \le 1 \qquad \ldots (4.1)$$

then the problem (1.1) has a unique solution on J.

Proof. Setting $\sup_{t \in f} |f(t, 0)| = M$ and $B_r = \{u \in PC(J, R) : \|u\|_{PC} \le R\}$, where

$$r \ge \frac{\frac{20+3q}{16\Gamma(q+1)} M + \frac{23}{16} \sum_{i=1}^{m} |J_i| + \frac{5}{4} \sum_{i=1}^{m} |I_i|}{1 - \frac{20+3q}{16\Gamma(q+1)}}$$

Define an operator $F : B_r \to PC(J, R)$ by

$$(Fu)(t) = \begin{cases} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))\, ds - \frac{1}{4\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s, u(s))\, ds \\ -\frac{(4t-1)}{16\Gamma(q-1)} \int_0^1 (1-s)^{q-2} f(s, u(s))\, ds - \frac{1}{16} \sum_{i=1}^{m} J_i [3 + 4(t - t_i)] - \frac{1}{4} \sum_{i=1}^{m} I_i \\ \text{for } t \in [0, t_1) \\ \vdots \\ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))\, ds - \frac{1}{4\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s, u(s))\, ds \\ -\frac{(4t-1)}{16\Gamma(q-1)} \int_0^1 (1-s)^{q-2} f(s, u(s))\, ds - \frac{1}{16} \sum_{i=1}^{m} J_i [3 + 4(t - t_i)] - \frac{1}{4} \sum_{i=1}^{m} I_i \\ + \sum_{i=1}^{k} I_i + \sum_{i=1}^{k} J_i (t - t_i), \\ \text{for } t \in (t_k, t_{k+1}), \\ k = 1, 2, \ldots, m. \end{cases} \qquad \ldots (4.2)$$

Clearly F is well defined

Step 1. We show that $FB_r \subset B_r$, in fact, for $u \in B_r,\, t \in J'$, we have

$$[(Fu)(t)] \le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\,|f(s, u(s))|\,ds + \frac{1}{4\Gamma(q)}\int_0^1 (1-s)^{q-1}\,|f(s,u(s))|\,ds$$

$$+\left|\frac{(4t-1)}{16\Gamma(q-1)}\right|\int_0^1 (1-s)^{q-2}\,|f(s, u(s))|\,ds + \frac{1}{16}\sum_{i=1}^{m} |J_i|\,|[3+4(t-t_i)]| + \frac{1}{4}\sum_{i=1}^{m} |I_i|$$

$$+\sum_{i=1}^{m} |I_i| + \sum_{i=1}^{k} |J_i|\,|(t-t_i)|.$$

$$\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\,|f(s, u(s)-f(s,0)|\,ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\,|f(s, 0)|\,ds$$

$$+\frac{1}{4\Gamma(q)}\int_0^1 (1-s)^{q-1}\,|f(s, u(s)-f(s,0)|\,ds + \frac{1}{4\Gamma(q)}\int_0^1 (1-s)^{q-1}\,|f(s, 0)|\,ds$$

$$+\frac{3}{16\Gamma(q-1)}\int_0^1 (1-s)^{q-2}\,|f(s, u(s)) - f(s,0)|\,ds$$

$$+\frac{3}{16\Gamma(q-1)}\int_0^1 (1-s)^{q-2}\,|f(s, 0)|\,ds + \frac{23}{16}\sum_{i=1}^{m} |J_i| + \frac{5}{4}\sum_{i=1}^{m} |I_i|$$

$$\le \frac{20+3q}{16\Gamma(q-1)}r + \frac{20+3q}{16\Gamma(q+1)}M + \frac{23}{16}\sum_{i=1}^{m} |J_i| + \frac{5}{4}\sum_{i=1}^{m} |I_i|$$

$$\le r.$$

Step 2. We show that F is a contraction mapping. For $u, v \in B_r$ and for each $t \in J'$, we get

$$|(Fu)(t)-(Fv)(t)| \le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\,|f(s, u(s)) - f(s, v(s))|\,ds$$

$$+\frac{1}{4\Gamma(q)}\int_0^1 (1-s)^{q-1}\,|f(s, u(s)) - f(s, v(s))|\,ds$$

$$+\left|\frac{(4t-1)}{16\Gamma(q-1)}\right|\int_0^1 (1-s)^{q-2}\,|f(s, u(s)) - f(s, v(s))|\,ds$$

$$\le \frac{20+3q}{16\Gamma(q+1)}\|u-v\|_{PC}.$$

Thus F is a contraction mapping on B_r due to condition (4.1).

By applying the well known Banach"s fixed point theorem we know thwt the operator F has a unique fixed point on B_r. therefore the problem (1.1) has a unique solution.

Our next result is based on the following well known Kransnoseelkii fixed point theorem.

Theorem 4.3. *Let M be a closed convex and nonempty subset of a Banach space X. Let P, Q be two operators such that*

(1) $Px + Qy \in M$ whenever $x, y \in M$.

(2) P is compact and continuous.

(3) Q is contraction mapping.

Then there exists *a* $z \in M$. such that $z = Pz + Qz$.

We introduce a comparison condition for nonlinear term

(H3) : There exists a $p \in (0, q-1)$ and a real function $\mu \in L^{1/p}(J, R^+)$ such that $|f(t, u)| \le \mu(t)$, for all $t \in J$ and $u \in R$.

Now we are ready to state and prove the following existence result.

Theorem 4.4. Assume (H1) and (H3) hold. Then the problem (1.1) has at least one solution on *J*.

Proof. Let us fix

$$r \le \|\mu\|_{L^{1/p}} \left[\frac{5}{4\Gamma(q)} \left(\frac{1-p}{q-p} \right)^{1-p} + \frac{3}{16\Gamma(q-1)} \left(\frac{1-p}{q-p-1} \right)^{1-p} \right] + \frac{23}{16} \sum_{i=1}^{m} |J_i| - \frac{5}{4} \sum_{i=1}^{m} |I|_i .$$

and consider $B_r = \{u \in P \subset (J.R) : \|u\|_{PC} \le r\}$. We define the operators P and Q on B_r as

$$(Pu)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))\, ds - \frac{1}{4\Gamma(q)} \int_0^1 (1-s)^{q-1} f(s, u(s))\, ds - \frac{(4t-1)}{16\Gamma(q-1)} \int_0^1 (1-s)^{q-2} f(s, u(s))\, ds$$

$$(Qu)(t) = -\frac{1}{16} \sum_{i=1}^{m} J_i [3 + 4(t - t_i)] - \frac{1}{4} \sum_{i=1}^{m} I_i - \sum_{i=1}^{k} I_i + \sum_{i=1}^{k} J_i (t - t_i).$$

For any $u, v \in B_r$ and $t \in J$ we find that

$$\begin{aligned} \|Pu + Qv\|_{PC} &\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \mu(s)\, ds + \frac{1}{4\Gamma(q)} \int_0^1 (1-s)^{q-1} \mu(s)\, ds \\ &\quad + \frac{3}{16\Gamma(q-1)} \int_0^1 (t-s)^{q-1} \mu(s)\, ds + \frac{23}{16} \sum_{i=1}^{m} |J_i| + \frac{5}{4} \sum_{i=1}^{m} |I_i| \\ &\le \|\mu\|_{L^{1/p}} \left[\frac{5}{4\Gamma(q)} \left(\frac{1-p}{q-p} \right)^{1-p} + \frac{3}{16\Gamma(q-1)} \left(\frac{1-p}{q-p-1} \right)^{1-p} \right] \\ &\quad + \frac{23}{16} \sum_{i=1}^{m} |J_i| + \frac{5}{4} \sum_{i=1}^{m} |I_i| \\ &\le r \end{aligned}$$

Thus $Pu + Qv \in B_r$, it is obviously Q is a contraction with constant zero. On the other hand the continuity of f implies that the operator P is continuous. Also P is uniformly bounded on B_r. Since

$$\| Pu \|_{PC} \leq \| \mu \|_{L^{1/p}} + \left[\frac{5}{4\Gamma(q)} \left(\frac{1-p}{q-p} \right)^{1-p} + \frac{1}{4\Gamma(q-1)} \left(\frac{1-p}{q-p-1} \right)^{1-p} \right]$$

$$+ \frac{23}{16} \sum_{i=1}^{m} |J_i| + \frac{5}{4} \sum_{i=1}^{m} |I_i|$$

$$\leq r$$

Now we need to prove compactness of operator P. Letting $\Omega = J \times B_r$, we can define $\sup_{(t,\, u) \in \Omega} |f(t, u)| = f_{\max}$ and consequently for any $t_k < \tau_1 < \tau_2 \leq t_{k+1}$ we have

$$|(Pu)(\tau_2) - (Pu)(\tau_1)|$$

$$= \left| \frac{1}{\Gamma(q)} \int_0^{\tau_2} (\tau_2 - s)^{q-1} f(s, u(s))\, ds - \frac{1}{\Gamma(q)} \int_0^{\tau_1} (\tau_1 - s)^{q-1} f(s, u(s))\, ds \right.$$

$$\left. + \frac{1}{4\Gamma(q-1)} (\tau_2 - \tau_1) \int_0^1 (1-s)^{q-2} f(s, u(s))\, ds \right|$$

$$\leq \left| \frac{1}{\Gamma(q)} \int_0^{\tau_1} [(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}] f(s, u(s)\, ds) \right.$$

$$\left. + \frac{1}{\Gamma(q)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{q-1} f(s, u(s))\, ds \right| + \left| \frac{(T_2 - T_1)}{4\Gamma(q-1)} \int_0^1 (1-s)^{q-2} f(s, u(s))\, ds \right|$$

$$\leq \frac{f_{\max}}{\Gamma(q)} \int_0^{\tau_1} [(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}]\, ds + \frac{f_{\max}}{\Gamma(q)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{q-1}\, ds + \frac{f_{\max}}{4\Gamma(q)(\tau_2 - \tau_1)}$$

$$= \frac{f_{\max}}{\Gamma(q+1)} \left[2(\tau_2 - \tau_1)^q + (\tau_2^q - \tau_1^q) + \frac{q(\tau_2 - \tau_1)}{4} \right]$$

which tends to zero as $\tau_2 \to \tau_1$. This yields that P is equicontinuous on interval $(t_k, t_{k+1}]$. So p is relatively compact on B_r.

By PC-type Arzela-Ascoli theorem (see theorem (2.10)). P is compact on B_r. Thus all the assumptions of theorem (4.4) implies that problem (1.1) has at least one solution on J. The proof is completed.

The third result is based on the following Schaefer's fixed point theorem.

Theorem 4.5. *Let X be a Banach space and $F : X \to X$ be a completely continuous operator. If the set $E(F) = \{y \in X : y = \lambda F y$ for some $\lambda \in [0, 1]\}$, is bounded then F has at least a fixed point.*

We introduce the following sublinear growth condition for nonlinear term.

(H3′) : There exist constant L > 0 and $\sigma \in \left(0, -\frac{1}{p}\right)$ *for some p (q – 2) + 1 > 0 with p > 1. such that* $\left| f(t, u) \le L(1 + |\mu|^{\sigma})\right|$ *for each* $t \in J$ *and all* $u \in R$.

Theorem 4.6. *Assume that (H1) and (H3′) hold. Then the problem (1.1) has at least one solution.*

Proof. Consider the operator $F : PC(J, R) \to PC(J, R)$ defined as (4.2). For the sake of convenience we subdivide the proof into several steps.

Step 1. F is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC(J, R)$ then for each $t \in J$ we have

$$\left|(Fu_n)(t) - (Fu)(t)\right| \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left| f(s, u_n(s)) - f(s, u(s)) \right| ds$$

$$+ \frac{1}{4\Gamma(q)} \int_0^1 (1-s)^{q-1} \left| f(s, u_n(s)) - f(s, u(s)) \right| ds$$

$$+ \left| \frac{(4t-1)}{16\Gamma(q-1)} \right| \int_0^1 (1-s)^{q-2} \left| f(s, u_n(s)) - f(s, u(s)) \right| ds$$

$$\le \frac{20 + 3q}{16\Gamma(q+1)} \| f(., u_n(.)) - f(., u(.)) \|_{PC}$$

Due to (H1), f is jointly continuous, then we have $\| Fu_n - Fu \|_{PC} \to 0$ as $n \to \infty$.

Step 2. F maps bounded sets into bounded sets in $PC(J, R)$. Indeed, it is enough to show that for any $\eta > 0$ there exists a $l > 0$ such that for each $u \in B_\eta = \{u \in PC(J, R) : \| u \|_{PC} \le \eta$, we have $\| Fu \|_{PC} \le l$.

For each $t \in J$, we get

$$\left|Fu(t)\right| \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left| f(s, u(s)) \right| ds + \frac{1}{4\Gamma(q)} \int_0^1 (1-s)^{q-1} \left| f(s, u(s)) \right| ds$$

$$+ \left| \frac{(4t-1)}{16\Gamma(q-1)} \right| \int_0^1 (1-s)^{q-2} \left| f(s, u(s)) \right| ds + \frac{1}{16} \sum_{i=1}^{m} \left| J_i \right| \left| 3 + 4(t - t_{1i}) \right|$$

$$+ \frac{1}{4} \sum_{i=1}^{m} \left| I_i \right| + \sum_{i=1}^{k} \left| J_i \right| \left| t - t_i \right|$$

$$\le \frac{20 + 3q}{16\Gamma(q-1)} L(1 + \eta^{\sigma}) + \frac{23}{13} \sum_{i=1}^{m} \left| J_i \right| + \frac{5}{4} \sum_{i=1}^{m} \left| I_i \right|,$$

which implies that

$$\| Fu \|_{PC} \le \frac{6 + q}{4\Gamma(q+1)} L(1 + \eta^{\sigma}) + \frac{7}{4} \sum_{i=1}^{m} \left| J_i \right| + \frac{3}{2} \sum_{i=1}^{m} \left| I_i \right| = l$$

Step 3. F maps bounded sets into equicontinuous sets of $PC\ (J,\ R)$ for interval $[0,\ t_1],\ 0 \le s_1 < s_2 \le t_1,\ u \in B_r,$ using (H3'), we have

$$\left|(Fu)(s_2)-(Fu)(s_1)\right| \le \frac{1}{\Gamma(q)}\int_0^{s_1} [(s_2-s_1)^{q-1}-(s_1-s)^{q-1}]\left|f(s,\ u(s))\right|ds$$

$$+\frac{1}{\Gamma(q)}\int_{s_1}^{s_2} (s_2-s)^{q-1}\left|f(s,\ u(s))\right|ds$$

$$+\frac{(s_2-s_{1)}}{4\Gamma(q-1)}\int_0^1 (1-s)^{q-2}\left|f(s,\ u(s))\right|ds+\frac{1}{16}\sum_{i=1}^{m} J_i\,[7(s_2-s_2)]$$

$$\le \frac{L(1-\eta^{\sigma})}{\Gamma(q)}\int_0^{s_1} [(s_2-s)^{q-1}-(s_1-s)^{q-1}]ds$$

$$+\frac{L(1-\eta^{\sigma})}{\Gamma(q)}\int_0^{s_1} (s_2-s)^{q-1}ds+\frac{L(1+\eta^{\sigma})}{4\Gamma(q)}-(s_2-s_1)+\frac{7}{16}\sum_{i=1}^{m} J_i(s_2-s_1)$$

$$\le \frac{L(1-\eta^{\sigma})}{\Gamma(q+1)}((s_2^q-s_1)^q+2(s_2-s_1)^q)\ +\left(\frac{L(1-\eta^{\sigma})}{4\Gamma(q)}+\frac{23}{16}\sum_{i=1}^{m} J_i\right)(s_1-s_1).$$

As $s_2 \to s_1,$ the right hand side of the above inequality tends to zero, therefore F is equicontinuous on interval $[0,\ t_1]$. In general, for the time interval (t_k, t_{k+1}), we similarly obtain the following inequality

$$\left\|(Fu)(s_2)-(Fu)(s_1)\right| \le \frac{L(1+\eta^{\sigma})}{\Gamma(q+1)}((s_2^q-s_1)^q+2(s_2-s_1)^q)$$

$$+\left(\frac{L(1+\eta^{\sigma})}{4\Gamma(q)}+\frac{23}{16}\sum_{i=1}^{m} J_i\right)(s_2-s_1) \to 0 \ \text{ as } s_2 \to s_1.$$

This yields that F is equicontinuous on the interval $(t_k,\ t_{k-1}]$. As a consequence of step 1– 3 together with PC-type Arzela-Ascoli theorem (theorem (2.10)) in case of $X = R$, we can conclude that $B_\eta \to B_\eta$ is continuous and completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set $E\ (F) = \{u \in PC\ (J,\ R) : u = \lambda Fu$, for some $\lambda \in (0,\ 1)\}$ is bounded. Let $u \in E\ (F)$, then $u = \lambda Fu$ for some $\lambda \in (0,\ 1)$. Without the loss of generality for some time interval $t \in (t_k,\ t_{k+1}]$,

$$\left|u(t)\right| \le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left|f(s,\ u(s))\right|ds+\frac{1}{4\Gamma(q)}\int_0^1 (1-s)^{q-1}\left|f(s,u(s))\right|ds$$

$$+\left|\frac{(4t-1)}{10\Gamma(q-1)}\right|\int_0^1 (1-s)^{q-2}\left|f(s,u(s))\right|ds+\frac{1}{16}\sum_{i=1}^{m} \left|J_i\right|\left|3+4(t-t_i)\right|$$

$$+\frac{1}{4}\sum_{i=1}^{m} \left|I_i\right|+\sum_{i=1}^{k} \left|I_i\right|+\sum_{i=1}^{k} \left|J_i\right|\left|(t-t_i)\right|$$

$$\leq \frac{L}{\Gamma(q)}\int_0^t (t-s)^{q-1}|u(s))|^{\sigma}\,ds + \frac{1}{4\Gamma(q)}\int_0^1 (1-s)^{q-1}|u(s))|^{\sigma}\,ds$$

$$+\frac{3L}{16\Gamma(q-1)}\int_0^1 (t-s)^{q-2}|u(s))|^{\sigma}\,ds + \frac{(20+3q)\,L}{16\Gamma(q+1)} + \frac{23}{16}\sum_{i=1}^{m} |J_i| + \frac{5}{4}\sum_{i=1}^{m} |I_i|.$$

by lemma (2.9), there exists a $M_k^* > 0$ such that

$$|u(t)| \leq M_k^*, \qquad t \in (t_k, t_{k+1}]$$

Set $M_k^* = \max\limits_{1\leq k\leq m}\{M_k^*\}$, thus for every $t \in J$, $|u(t)| \leq M^*$ which yields that $\|u\|_{PC} \leq M^*$.

This shows that $E(F)$ is bounded.

As the consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1.1). The proof is completed.

Examples

In this section we give some illustrations to prove usefulness of main result.

Example 5.1. *Let us consider the first impulsive anti-periodic problem.*

$$\left.\begin{array}{ll} {}^cD_t^{5/2}u(t) = t, & t \in [0,1]\setminus\left\{\frac{1}{3}\right\} \\ \Delta u\left(\frac{1}{3}\right) = \frac{1}{2}, & \Delta u'\left(\frac{1}{3}\right) = \frac{1}{2}, \\ 3u(0) = -u(1), & 3u'(0) = -u'(1). \end{array}\right\} \qquad \ldots (5.1)$$

Set $f(t, u) = t$, $(t, u) \in [0, 1] \times R$. Obviously f is a nonlinear D-contraction on u. One can arrive at following inequality

$$\frac{20+3q}{16\Gamma(q+1)} = \frac{20+3\left(\frac{5}{2}\right)}{16\Gamma\left(\frac{7}{2}\right)} \approx 0.51717379 < 1.$$

Thus the assumptions in Theorem (4.2) are satisfied. Hence the problem (5.1) has at least one solution on [0, 1],

Example 5.2. *Let us consider the second impulsive anti-periodic problem.*

$$\left.\begin{array}{ll} {}^cD_t^{3/2}u(t) = \dfrac{e^t\,|u(t)|}{(t+1)^2\,(1+|u(t)|)}, & t \in [0,1]\setminus\left\{\frac{1}{5}\right\} \\ \Delta u\left(\frac{1}{5}\right) = I_1, & \Delta u'\left(\frac{1}{5}\right) = I_1, \\ 3u(0) = -u(1), & 3u'(0) = -u'(1). \end{array}\right\} \qquad \ldots (5.2)$$

Set $f(t, u) = \dfrac{e^t\,|u(t)|}{(t+1)^2\,(1+|u(t)|)}$, $(t, u) \in [0, 1] \times R^+$ and $t \in [0, 1]$. Obviously, $|f(t, u)| \le \dfrac{e^t}{(t+1)^2}$. Denote $q = \dfrac{3}{2}$, $p = \dfrac{1}{4}$, and $\mu(t) = \dfrac{e^t}{(1+1)^2} \in L^4([0, 1], R^+)$. Thus all the assumptions of Theorem (4.4) are satisfied. Hence the problem (5.2) has at least one solution on [0, 1].

Example 5.3. *Let us consider the third impulsive anti-periodic problem.*

$$\left.\begin{array}{ll} {}^cD_t^{3/2}u(t) = \dfrac{|u(t)^{1/5}|}{(1+e^t)^2\,(1+|u(t)|)}, & t \in [0,1] \setminus \left\{\dfrac{1}{3}\right\} \\ \Delta u\left(\dfrac{1}{3}\right) = I_1, & \Delta u'\left(\dfrac{1}{3}\right) = J_1, \\ 3u(0) = -u(1), & 3u'(0) = -u'(1). \end{array}\right\} \quad \dots (5.3)$$

Set $f(t, u) = \dfrac{|u(t)^{1/5}|}{(1+e^t)^2\,(1+|u(t)|)}$, $(t, u) \in [0, 1] \times R^+$

and $t \in [0, 1]$, $|f(t, u)| \le \dfrac{1}{2}(1 + |u|^{1/5})$.

Thus all the assumptions of Theorem (4.6) are satisfied. Hence the problem (5.3) has at least one solution on [0.1].

A large number of fractional models on theoretical physics have been reported by Tarasov [28]. In order to show that our theory results can be applied to solve some physical models, we turn to consider the following generalized impulsive spring-pot model with anti-periodic boundary value conditions.

$$\left.\begin{array}{ll} {}^cD_t^{3/5}\upsilon(t) = -\upsilon(t) + \sigma(t), & t \in [0,1] \setminus \left\{\dfrac{1}{2}\right\} \\ \Delta\upsilon\left(\dfrac{1^+}{2}\right) = \upsilon\left(\dfrac{1^-}{2}\right) + I_1, & \Delta\upsilon'\left(\dfrac{1^+}{2}\right) = \upsilon'\left(\dfrac{1^-}{2}\right) + J_1, \quad I_1, J_1 > 0 \\ \upsilon(0) = -\upsilon(1), & \upsilon'(0) = -\upsilon'(1). \end{array}\right\} \quad \dots (5.4)$$

where σ is stress, υ is strain, $\dfrac{1}{2}$ is possible impulsive perturbed time, and I_1, J_1 are impulsive perturbed constants. Similar to discussion in example (5.1), the results in above section can be used to solve the model (5.2).

Conclusion

An anti-periodic boundary value problem for impulsive fractional differential equations involving Caputo fractional derivative has been studied. A better formula and definition of solutions for such problem is introduced. Many existence theorem of solutions are presented under some general and different mixed conditions such as nonlinear *D*-contraction condition, comparison condition, sublinear growth condition via hybrid singular type Gronwell inequality and fixed point technique.

Conflict of interest

The authors declare that they have no conflict of interest.

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