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GENERALISED RELATION AMONG UPPER BOUNDS IN A SPECIAL CASE BY TAYLOR'S SERIES

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In this paper, it is aimed to investigate a generalized relation among upper bounds in a Special Case by Taylor's Series with a specified length of the interval 'h'. A common relation among upper bounds is identified.

KEYWORDS : Maxima & Minima, upper bounds, Series, Taylor's Theorem.

AMS Subject Classification (2000): 65L80, 65L05, 41A58

INTRODUCTION

It is a known fact that Taylor's series is an effective and efficient tool to express a function as an infinite series. Taylor introduced this concept in 1715. Later the Taylors series was concentrated at origin as a special case by Meclarence in 18th century. In a Taylor's series, a finite number of terms may approximate a function with minimized errors. Taylor's polynomial is constituted by some initial terms of Taylor's series. In a single variable system, Taylor's series consists of values of functional derivatives at a single quantity. Many methods are often available for calculating Taylor's series in a large number of functions. In the recent development of advanced science, computer algebra computational systems are considered to calculate Taylor's series in a fast manner.

Many applications of Taylor's series exist in multi various technical fields. Taylor's polynomials are very useful to approximate single value. It has it's own significance in 'the zeros of Bessel's function'. It is utilized to approximate a definite integral. On the other hand Taylor's polynomials are also helpful to conjecture the value of limits.

Few advantages of Taylor's series are identified in complicated computations. In this connection, it needs the explicit form of derivatives of the function. Even though the system has higher order, Taylor's series is still valid. In many cases, the representation of Taylor's series simplifies and clarifies many mathematical calculations. Mostly the partial sums of Taylor series can be utilized for calculating approximation of entire function even many terms occurred. In any computation, there is a chance of possibility to get an Error. The Error should be minimized to obtain a stabilized system. Minimized Error will lead to strengthen the system. Error ($E_n(x)$) can be calculated with the following formula

$$|E_n(x)| = \frac{1}{n!}\int_a^x (n-t)^n f^{(n+1)}(t)\,dt \qquad \dots(1)$$

MATERIALS AND METHODS

Taylor's Theorem : - Let $(n-1)^{\text{th}}$ derivative of f. *i.e.* $f^{(n-1)}$ is continuous in $[a, a+h]$. The n^{th} derivative of f. *i.e.* $f^{(n)}$ exists in $(a, a+h)$. Then there exists at least one positive number θ lying between 0 and 1 such that

$$f(a+h) = f(a) + h\frac{f'(a)}{1!} + h^2\frac{f''(a)}{2!} + h^3\frac{f'''(a)}{3!} + \cdots + h^{n-1}\frac{f^{(n-1)}(a)}{(n-1)!} + R_n \quad \dots(2)$$

where $R_n = \dfrac{h^n(1-\theta)^{(n-k)}}{(n-1)!\,k} f^{(n)}(a+\theta h)$ and $0 < \theta < 1$, here k be a positive integer

Taylor's remainder after n terms R_n occurred in various concepts as below

1. Scholmich and Roche: $R_n = \dfrac{h^n(1-\theta)^{(n-k)}}{(n-1)!\,k} f^{(n)}(a+\theta h)$

2. Cauchy : $k = 1$; $R_n = \dfrac{h^n(1-\theta)^{(n-1)}}{(n-1)!} f^{(n)}(a+\theta h)$

3. Lagranges : $k = n$; $R_n = \dfrac{h^n}{(n)!} f^{(n)}(a+\theta h)$

Then $\quad f(x) = P_n(x) + R_n(x)$ or $|R_n(x)| = |f(x) - P_n(x)|$

Here $R_n(x)$ acts as an error. It is further denoted by $E_n(x)$

i.e., $\quad |E_n(x)| = |P_n(x) - f(x)|$.

If $E_n(x)$ is small which tends towards zero, the polynomial $P_n(x)$ becomes the function $f(x)$.

It can be shown as in the following figure

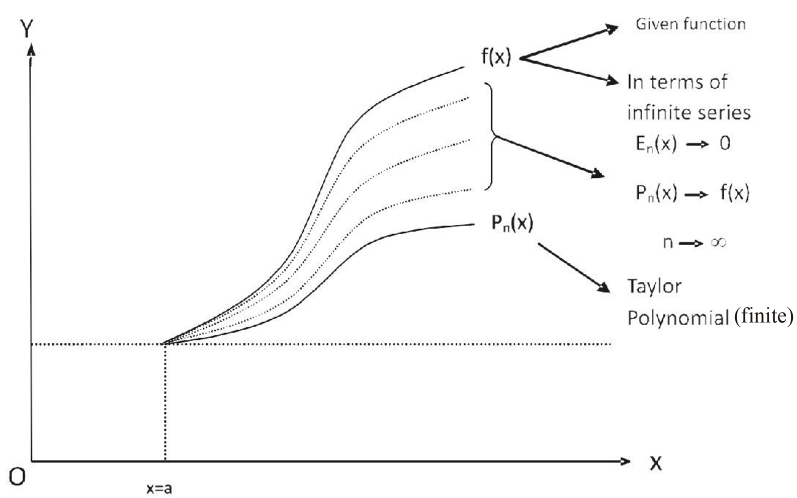

Fig. 1

If we consider $a = 0$ in (2), then the Taylor's formula at origin (0, 0) reduces as

$$f(h) = f(0) + h\frac{f'(0)}{1!} + h^2\frac{f''(0)}{2!} + h^3\frac{f'''(0)}{3!} + \cdots + h^{n-1}\frac{f^{(n-1)}(0)}{(n-1)!} + h^n\frac{f^{(n)}(0)}{n!} \quad \ldots(3)$$

It is known as Maclaurnin's formula. In (2) as $R_n \to 0$ as $n \to \infty$,

Then $f(x)$ is represented by infinite series in powers of $(x - a)$ and is given by

$$f(x) = f(a) + (x-a)\frac{f'(a)}{1!} + (x-a)^2\frac{f''(a)}{2!} + \cdots + (x-a)^n\frac{f^{(n)}(a)}{n!} + \cdots$$

The above power series is known as Taylor's series which can be represented as

$$f(x) = f(a) + \sum_{n=1}^{\infty}\frac{(x-a)^n}{n!}f^{(n)}(a)$$

Basic idea

Walter Ruddin [10] discussed the concept of Taylor's theorem clearly in "Principles of Mathematical Analysis". In this connection, he specified a problem [7] in the exercise which was focused on the relation between the upper bounds.

With this basic idea, the authors aim to investigate the relation among all possible existing upper bounds after few stages of computations.

Main relation among upper bounds

Suppose $a \in R$, $f^{(n)}$ exists on (a, ∞) and $M_0, M_1, M_2, \ldots, M_n$ are upper bounds of $|f(x)|, |f'(x)|, |f''(x)|, \ldots, |f^{(n)}(x)|$ respectively in the specified interval .Then we can show that there exists a positive real root in (a,∞) and the generalized relation among upper bounds is derived as $9M_1 \le 14M_0 + 6M_2 + 3M_3 + M_4 + 0.2M_5 + 0.04M_6 + 0.008M_7 + 0.0009M_8 + 0.0002M_9 + 0.00003M_{10} + 0.000003M_{11} + 0.0000003M_{12} + 0.000000003M_{13} + \ldots$,

Proof : By hypothesis, f has n derivatives on (a,∞), *i.e.*, $|f^{(n)}(x)|$ exists on (a, ∞)

By Taylor's theorem,

$$f(\beta) = p(\beta) + \frac{f^{(n)}(\gamma)}{n!} \text{ in which } \gamma \in (\alpha, \beta) \text{ where } a < \alpha < \gamma < \beta < \infty$$

Let $\alpha = x, \beta = x + h$ and $h > 0$ and $p(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\gamma)}{k!}(t-\alpha)^k$

By Taylor's series, $f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots,$

It can be simplified as $|hf'(x)| \le |f(x+h)| + |f(x)| + \left|\frac{h^2}{2!}f''(x)\right| + \left|\frac{h^3}{3!}f'''(x)\right| + \cdots,$

Let $M_0, M_1, M_2, \ldots, M_n$ are upper bounds of $|f(x)|, |f'(x)|, |f''(x)|, \ldots, |f^{(n)}(x)|$

Then $M_1 \leq \frac{2}{h} M_0 + \frac{h}{2!} M_2 + \frac{h^2}{3!} M_3 + \frac{h^3}{4!} M_4 + \frac{h^4}{5!} M_5 + \frac{h^5}{6!} M_6 + \frac{h^6}{7!} M_7 + \frac{h^7}{8!} M_8 + \frac{h^8}{9!} M_9$

$+ \frac{h^9}{10!} M_{10} + \frac{h^{10}}{11!} M_{11} + \frac{h^{11}}{12!} M_{12} + \frac{h^{12}}{13!} M_{13} + \frac{h^{13}}{14!} M_{14} + \frac{h^{14}}{15!} M_{15} + \frac{h^{15}}{16!} M_{16} + \ldots,$

It is a known fact that $|f'(x)| \leq$ least value of $\varphi(h)$

$$= \frac{2}{h} M_0 + \frac{h}{2!} M_2 + \frac{h^2}{3!} M_3 + \frac{h^3}{4!} M_4 + \frac{h^4}{5!} M_5 + \frac{h^5}{6!} M_6 + \frac{h^6}{7!} M_7 + \frac{h^7}{8!} M_8 + \cdots,$$

The function $\varphi(h)$ will attain minimum or maximum only when $\varphi'(h) = 0$.

i.e. $\sum_{k=1}^{n} \frac{(n-k)\, h^{(n-(k-1))}}{(n-(k-1))!} M_{(n-(k-1))} - 2M_0 = 0$, where $M = \max\{M_0, M_2, M_3, M_4, \ldots, M_n\}$

For *n* = 2, The Equation is $\frac{h^2}{2!} - 2 = 0$, $h^2 - 4 = 0$

From (2),

$f(x+h) = p(x+h) + \frac{f^{(2)}(\gamma)}{2!}(x+h-x)^2$ *i.e.*, $f(x+h) = p(x+h) + \frac{f^{(2)}(\gamma)}{2!}(h)^2$,

The roots are obtained as 2, 2

In this case, the relation among upper bounds M_1, M_0, M_2 is obtained as $M_1 \leq M_0 + M_2$

By extending the same concept to the next level,

For *n* = 3, The Equation is $2h^3 + 3h^2 - 12 = 0$

The roots are traced as 1.431, –1.4654 + 1.4304i, – 1.4654 – 1.4304i.

Here, h should be considered as a positive value.

From (2), $M_1 \leq \frac{2}{h} M_0 + \frac{h}{2!} M_2 + \frac{h^2}{3!} M_3$

By computing further, the relation among upper bounds is obtained as

$$9M_1 \leq 13M_0 + 6M_2 + 3M_3$$

For *n* = 4, The Equation is $3h^4 + 8h^3 + 12h^2 - 48 = 0$

The roots are obtained as 1.3159, –2.2712, – 0.8557 + 2.1498i, – 0.8557 – 2.1498i

Here, h should be calculated as a positive value.

From (2), $M_1 \leq \frac{2}{h} M_0 + \frac{h}{2!} M_2 + \frac{h^2}{3!} M_3 + \frac{h^3}{4!} M_4$

By calculating in the same manner, the relation among upper bounds is identified as

$$9M_1 \leq 14M_0 + 6M_2 + 3M_3 + M_4$$

For *n* = 5, The Equation is $4h^5 + 15h^4 + 40h^3 + 60h^2 - 240 = 0$

The roots are obtained as

1.287, – 2.2009 + 1.2393i, – 2.2009 – 1.2393i, – 0.3177 + 2.6849i, – 0.3177 – 2.6849i

From (2), $$M_1 \leq \frac{2}{h} M_0 + \frac{h}{2!}M_2 + \frac{h^2}{3!}M_3 + \frac{h^3}{4!}M_4 + \frac{h^4}{5!}M_5$$

By computing further, the relation among upper bounds is obtained as

$$9M_1 \leq 14M_0 + 6M_2 + 3M_3 + M_4 + 0.2M_5$$

For $n = 6$, The Equation is $5h^6 + 24h^5 + 90h^4 + 240h^3 + 360h^2 - 1440 = 0$

The roots are obtained as

1.2789, –2.6493, $0.3491 + 3.1561i$, $0.3491 - 3.1561i$, $-1.6639 + 2.0631i$, $-1.6639 - 2.0631i$.

From (2), $$M_1 \leq \frac{2}{h} < M_0 + \frac{h}{2!}M_2 + \frac{h^2}{3!}M_3 + \frac{h^3}{4!}M_4 + \frac{h^4}{5!}M_5 + \frac{h^5}{6!}M_6$$

By calculating in the same manner, the relation among upper bounds is identified as

$$9M_1 \leq 14M_0 + 6M_2 + 3M_3 + M_4 + 0.2M_5 + 0.04M_6$$

For $n = 7$, the equation is $6h^7 + 35h^6 + 168h^5 + 630h^4 + 1680h^3 + 2520h^2 - 10080 = 0$

The roots are obtained as

1.2788, $0.7207 + 3.6991i$, $0.7207 - 3.6991i$, $-2.8980 + 1.1850i$, $-2.8980 - 1.1850i$, $-1.3787 + 2.7451i$, $-1.3787 - 2.7451i$,

From (2), $$M_1 \leq \frac{2}{h}M_0 + \frac{h}{2!}M_2 + \frac{h^2}{3!}M_3 + \frac{h^3}{4!}M_4 + \frac{h^4}{5!}M_5 + \frac{h^5}{6!}M_6 + \frac{h^6}{7!}M_7$$

By computing further, the relation among upper bounds is obtained as

$$9M_1 \leq 14M_0 + 6M_2 + 3M_3 + M_4 + 0.2M_5 + 0.04M_6 + 0.008M_7$$

For $n = 8$, The Equation is

$$7h^8 + 48h^7 + 280h^6 + 1344h^5 + 5040h^4 + 13440h^3 + 20160h^2 - 80640 = 0$$

The roots are obtained as 1.2785, – 3.4676, $1.2981 + 4.2149i$, $1.2981 - 4.2149i$, $-2.6355 + 2.1595i$, $-2.6355 - 2.1595i$, $-0.9967 + 3.2426i$, $-0.9967 - 3.2426i$,

From (2), $$M_1 \leq \frac{2}{h}M_0 + \frac{h}{2!}M_2 + \frac{h^2}{3!}M_3 + \frac{h^3}{4!}M_4 + \frac{h^4}{5!}M_5 + \frac{h^5}{6!}M_6 + \frac{h^6}{7!}M_7 + \frac{h^7}{8!}M_8$$

By calculating in the same manner, the relation among upper bounds is identified as

$$9M_1 \leq 14M_0 + 6M_2 + 3M_3 + M_4 + 0.2M_5 + 0.04M_6 + 0.008M_7 + 0.0009M_8$$

For $n = 9$, The Equation is

$$8h^9 + 63h^8 + 432h^7 + 2520h^6 + 12096h^5 + 45360h^4 + 120960h^3 + 181440h^2 - 725760 = 0$$

The roots are obtained as

1.2785, $1.9248 + 4.7134i$, $1.9248 - 4.7134i$, $-3.5996 + 1.1674i$, $-3.5996 - 1.1674i$, $-2.2185 + 2.9017i$, $-2.2185 - 2.9017i$, $-0.6835 + 3.7231i$, $-0.6835 - 3.7231i$,

From (2), $$M_1 \leq \frac{2}{2!}M_2 + \frac{h^2}{3!}M_3 + \frac{h^3}{4!}M_4 + \frac{h^4}{5!}M_5 + \frac{h^5}{6!}M_6 + \frac{h^6}{7!}M_7 + \frac{h^7}{8!}M_8 + \frac{h^8}{9!}M_9$$

By computing further, the relation among upper bounds is obtained as

$$9M_1 \leq 14M_0 + 6M_2 + 3M_3 + M_4 + 0.2M_5 + 0.04M_6 + 0.008M_7 + 0.0009M_8 + 0.0002M_9$$

For *n* = 10, The Equation is

$$9h^{10} + 80h^9 + 630h^8 + 4320h^7 + 25200h^6 + 120960h^5 + 453600h^4 + 1209600h^3 + 1814400h^2 - 7257600 = 0$$

The roots are obtained as 1.2785, – 4.1322, 2.5874 + 5.1849i, 2.5874 – 5.1849i, –3.4351 + 2.2067i, – 3.4351 – 2.2067i, – 0.3928 + 4.2679i, – 0.3928 – 4.2679i, – 1.7771 + 3.4188i, –1.7771 – 3.4188i,

From (2),

$$M_1 \leq \frac{2}{h}M_0 + \frac{2}{2!}M_2 + \frac{h^2}{3!}M_3 + \frac{h^3}{4!}M_4 + \frac{h^4}{5!}M_5 + \frac{h^5}{6!}M_6 + \frac{h^6}{7!}M_7 + \frac{h^7}{8!}M_8 + \frac{h^8}{9!}M_9 + \frac{h^9}{10!}$$

By calculating in the same manner, the relation among upper bounds is identified as

$$9M_1 \leq 14M_0 + 6M_2 + 3M_3 + M_4 + 0.2M_5 + 0.04M_6 + 0.008M_7 + 0.0009M_8 + 0.0002M_9 + 0.00003M_{10}$$

For *n* = 11, The Equation is

$$10h^{11} + 99h^{10} + 880h^9 + 6930h^8 + 47520h^7 + 277200h^6 + 1330560h^5 + 4989600h^4 + 13305600h^3 + 19958400h^2 - 79833600 = 0$$

The roots are obtained as 1.2785, 3.2739 + 5.6299i, 3.2739 – 5.6299i, – 0.0312 + 4.8822i, – 0.0312 – 4.8822i, – 4.3093 + 1.1612i, – 4.3093 – 1.1612i, – 3.0741 + 3.0870i, – 3.0741 – 3.0870i, – 1.4485 + 3.7736i, – 1.4485 – 3.7736i,

From (2),

$$M_1 \leq \frac{2}{h}M_0 + \frac{h}{2!}M_2 + \frac{h^2}{3!}M_3 + \frac{h^3}{4!}M_4 + \frac{h^4}{5!}M_5 + \frac{h^5}{6!}M_6 + \frac{h^6}{7!}M_7 + \frac{h^7}{8!}M_8 + \frac{h^8}{9!}M_9 + \frac{h^9}{10!} + \frac{h^{10}}{11!}$$

By computing further, the relation among upper bounds is obtained as

$$9M_1 \leq 14M_0 + 6M_2 + 3M_3 + M_4 + 0.2M_5 + 0.04M_6 + 0.008M_7 + 0.0009M_8 + 0.0002M_9 + 0.00003M_{10} + 0.000003M_{11}$$

<u>For n=12,</u> The Equation is

$$11h^{12} + 120h^{11} + 1188h^{10} + 10560h^9 + 83160h^8 + 570240h^7 + 3326400h^6 + 15966720h^5 + 59875200h^4 + 159667200h^3 + 239500800h^2 - 958003200 = 0$$

The roots are obtained as 1.2785, -4.8162, 3.9773+ 6.0523i, 3.9773- 6.0523i, 0.4287+ 5.4888i, 0.4287- 5.4888i, -4.2257+ 2.2394i, -4.2257- 2.2394i, -2.5606+3.7956i, -2.5606- 3.7956i, -1.3054+4.1051i, -1.3054- 4.1051i,

From (2),

$$M_1 \leq \frac{2}{h}M_0 + \frac{2}{2!}M_2 + \frac{h^2}{3!}M_3 + \frac{h^3}{4!}M_4 + \frac{h^4}{5!}M_5 + \frac{h^5}{6!}M_6 + \frac{h^6}{7!}M_7 + \frac{h^7}{8!}M_8 + \frac{h^8}{9!}M_9 + \frac{h^9}{10!}M_{10} + \frac{h^{10}}{11!}M_{11} + \frac{h^{11}}{12!}M_{12}$$

By calculating in the same manner, the relation among upper bounds is identified as

$$9M_1 \leq 14M_0 + 6M_2 + 3M_3 + M_4 + 0.2M_5 + 0.04M_6 + 0.008M_7 + 0.0009M_8 + 0.0002M_9 + 0.00003M_{10} + 0.000003M_{11} + 0.0000003M_{12}$$

For *n* = 13, The Equation is

$$12h^{13} + 143h^{12} + 1560h^{11} + 15444h^{10} + 137280h^9 + 1081080h^8 + 7413120h^7 + 43243200h^6 + 207567360h^5 + 778377600h^4 + 2075673600h^3 + 3113510400h^2 - 12454041600 = 0$$

The roots are obtained as 1.2785, $4.6942 + 6.4558i$, $4.6942 - 6.4558i$, $0.9461 + 6.0533i$, $0.9461 - 6.0533i$, $-5.0250 + 1.1584i$, $-5.0250 - 1.1584i$, $-3.9548 + 3.2105i$, $-3.9548 - 3.2105i$, $-1.4142 + 4.6663i$, $-1.4142 - 4.6663i$, $-1.8439 + 4.1583i$, $-1.8439 - 4.1583i$,

From (2),

$$M_1 \le \frac{2}{h}M_0 + \frac{2}{2!}M_2 + \frac{h^2}{3!}M_3 + \frac{h^3}{4!}M_4 + \frac{h^4}{5!}M_5 + \frac{h^5}{6!}M_6 + \frac{h^6}{7!}M_7 + \frac{h^7}{8!}M_8 + \frac{h^8}{9!}M_9 + \frac{h^9}{10!}M_{10} + \frac{h^{10}}{11!}M_{11} + \frac{h^{11}}{12!}M_{12} + \frac{h^{12}}{13!}M_{13}$$

By computing further, the relation among upper bounds is obtained as

$$9M_1 \le 14M_0 + 6M_2 + 3M_3 + M_4 + 0.2M_5 + 0.04M_6 + 0.008M_7 + 0.0009M_8 + 0.0002M_9 + 0.00003M_{10} + 0.000003M_{11} + 0.0000003M_{12} + 0.000000003M_{13}$$

Conclusions

A generalized relation among upper bounds is established in a Special Case by Taylor's Series with a specified length of the interval '*h*'.

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